Application of Lie Transform Techniques for simulation of a charged particle beam

Mathieu Lutz

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Abstract

We study a Lie Transform method for a charged beam under the action of a radial external electric field. The aim of the Lie transform method that is used here is to construct a change of variable which transforms the 2D kinetic problem into a 1D problem. This reduces the dimensionality of the problem and make it easier to solve numerically. After applying the Lie transform method, we truncate the expression of the characteristics of the Vlasov equation and the expression of the Poisson equation in the Lie coordinate system and we develop a numerical method for solving the truncated model and we study its efficiency for the simulation of long time beam evolution.

Keywords: Vlasov-Poisson system, kinetic equations, homogenization, Lie transform, Lie transform PIC method, gyrokinetic.

1 Introduction

In the same spirit of [4], we will consider non-relativistic long and thin beams. Within the general framework, if we neglect the collisions between particles, the particle density is obtained by solving a Vlasov Maxwell system of equations. Here, in addition to consider a long and thin beam, we will consider a beam satisfying the following assumptions :

- The beam is steady-state: all partial derivatives with respect to time vanish.
- The beam is long and thin.
- The beam is propagating at constant velocity v_b along the propagation axis z.
- The beam is sufficiently long so that longitudinal self-consistent forces can be neglected.
- The external electric field is supposed to be independent of the time.
- The beam is axisymmetric.
- The initial distribution f_0 is concentrated in angular momentum.

*Université de Strasbourg, IRMA, 7 rue René Descartes, F-67084 Strasbourg Cedex, France & Projet INRIA Calvi.

Under the five first assumptions, the $3D$ Vlasov-Maxwell system reduces itself to a $2D$ Vlasov-Poisson system in which the variable t does not represent, from a physical point of view, a time variable, but rather the longitudinal coordinate. The details about the derivation of this model can be found in [3]. Moreover, under all these assumptions it reduces even to a $1D$ axisymmetric Vlasov-Poisson system of the form

$$\frac{\partial f_\varepsilon}{\partial t} + \frac{v_r}{\varepsilon}\frac{\partial f_\varepsilon}{\partial r} + \left(E^\varepsilon - \frac{r}{\varepsilon}\right)\frac{\partial f_\varepsilon}{\partial v_r} = 0, \tag{1.1}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_\varepsilon}{\partial r}\right) = \rho_\varepsilon(t,r), \quad E_\varepsilon = -\frac{\partial \phi^\varepsilon}{\partial r}, \tag{1.2}$$

$$\rho_\varepsilon(t,r) = \int_{\mathbb{R}} f_\varepsilon(t,r,v_r)\,dv_r, \tag{1.3}$$

$$E_\varepsilon(t, r=0) = 0, \quad \phi_\varepsilon(t, r=0) = 0, \tag{1.4}$$

$$f_\varepsilon(t=0, r, v_r) = f_0(r, v_r), \tag{1.5}$$

where $r \geq 0$ is the radial component of the projection of the position vector in the transverse plane to the propagation direction, $v_r \in \mathbb{R}$ is the projection of the transverse velocity in the transverse plan to the propagation direction, ε is the ratio between the characteristic transverse radius of the beam and the characteristic longitudinal length of the beam, $f_\varepsilon = f_\varepsilon(t,r,v_r)$ is the distribution function of the particles, $E_\varepsilon = E_\varepsilon(r,t)$ is the radial part of the transverse self-consistent electric field, and $-\frac{r}{\varepsilon}$ is the strong transverse external electric field. This system is naturally defined for $r \geq 0$ but we can extend it to $r \in \mathbb{R}$ by using the conventions $f_\varepsilon(t,r,v_r) = f_\varepsilon(t,-r,-v_r)$ and $E_\varepsilon(t,r) = -E_\varepsilon(t,-r)$. Details about the derivation of this model can be found in [4]. Moreover, in the same way as in [4] we will consider initial conditions for which the beam is confined. Such initial conditions can be found by solving envelope equations (see [3] for details about the obtention of such initial conditions).

The characteristics of (1.1) are given by

$$\frac{\partial \mathbf{R}^\varepsilon}{\partial t} = \frac{\mathbf{V}^\varepsilon_{\mathbf{r}}}{\varepsilon}, \qquad \mathbf{R}^\varepsilon(0,r,v_r) = r, \tag{1.6}$$

$$\frac{\partial \mathbf{V}^\varepsilon_r}{\partial t} = -\frac{\mathbf{R}^\varepsilon}{\varepsilon} + E_\varepsilon(\mathbf{R}^\varepsilon, t), \qquad \mathbf{V_r}^\varepsilon(0,r,v_r) = v_r. \tag{1.7}$$

Setting

$$H_\varepsilon(r, v_r, t) = \frac{v_r^2 + r^2}{2\varepsilon} + \phi_\varepsilon(r,t), \tag{1.8}$$

dynamical system (1.6)-(1.7) becomes:

$$\frac{\partial \mathbf{R}^\varepsilon}{\partial t} = \partial_{v_r} H_\varepsilon(\mathbf{R}^\varepsilon, \mathbf{V}^\varepsilon_r, t), \qquad \mathbf{R}^\varepsilon(0,r,v_r) = r, \tag{1.9}$$

$$\frac{\partial \mathbf{V}^\varepsilon_r}{\partial t} = -\partial_r H_\varepsilon(\mathbf{R}^\varepsilon, \mathbf{V}^\varepsilon_r, t), \qquad \mathbf{V_r}^\varepsilon(0,r,v_r) = v_r. \tag{1.10}$$

Consequently the dynamical system that gives the characteristics is Hamiltonian.

Furthermore, dynamical system (1.6)-(1.7) corresponds to a perturbation of dynamical system

$$\frac{\partial \mathbf{R}^{\varepsilon}_{\mathbf{Un}}}{\partial t} = \frac{\mathbf{V}^{\varepsilon}_{\mathbf{r},\mathbf{Un}}}{\varepsilon}, \qquad \mathbf{R}^{\varepsilon}_{\mathbf{Un}}\left(0, r, v_r\right) = r, \tag{1.11}$$

$$\frac{\partial \mathbf{V}^{\varepsilon}_{r,\mathbf{Un}}}{\partial t} = -\frac{\mathbf{R}^{\varepsilon}_{\mathbf{Un}}}{\varepsilon}, \qquad \mathbf{V}^{\varepsilon}_{r,\mathbf{Un}}\left(0, r, v_r\right) = v_r. \tag{1.12}$$

In other words the Hamiltonian function (1.8) is a perturbation of the Hamiltonian function

$$H^{\mathbf{Un}}_{\varepsilon}(r, v_r, t) = \frac{v_r^2 + r^2}{2\varepsilon}, \tag{1.13}$$

associated to the dynamical system (1.11)-(1.12).

A well adapted coordinate system for the study of the dynamical system (1.11)-(1.12) is the (μ, θ) coordinate system defined by

$$\mu = \frac{r^2 + v_r^2}{2}. \tag{1.14}$$

and

$$r = \sqrt{2\mu}\cos\left(\theta\right), \tag{1.15}$$

$$v_r = \sqrt{2\mu}\sin\left(\theta\right). \tag{1.16}$$

Indeed, in this coordinate system the dynamical system (1.11)-(1.12) reads:

$$\frac{\partial \mathfrak{Mu}^{\varepsilon}_{\mathbf{Un}}}{\partial t} = 0, \qquad \mathfrak{Mu}^{\varepsilon}_{\mathbf{Un}}\left(0, \mu, \theta\right) = \mu, \tag{1.17}$$

$$\frac{\partial \mathbf{\Theta}^{\varepsilon}_{\mathbf{Un}}}{\partial t} = -\frac{1}{\varepsilon}, \qquad \mathbf{\Theta}^{\varepsilon}_{\mathbf{Un}}\left(0, \mu, \theta\right) = \theta. \tag{1.18}$$

As a consequence, solving this dynamical system in the new system of coordinates, reduces to find a trajectory in $\mathbb{R}$, in place of a trajectory in $\mathbb{R}^2$ when it is solved in the original system of coordinates.

Under the same change of coordinates, the Hamiltonian function associated to dynamical system (1.6)-(1.7) becomes:

$$\bar{H}_{\varepsilon}\left(\mu, \theta, t\right) = \frac{\mu}{\varepsilon} + \phi_{\varepsilon}\left(\sqrt{2\mu}\cos\left(\theta\right), t\right), \tag{1.19}$$

and the dynamical system (1.6)-(1.7) reads:

$$\frac{\partial \mathfrak{Mu}^{\varepsilon}}{\partial t} = \sqrt{2\mathfrak{Mu}^{\varepsilon}}\sin\left(\mathbf{\Theta}^{\varepsilon}\right) E_{\varepsilon}\left(\sqrt{2\mathfrak{Mu}^{\varepsilon}}\cos\left(\mathbf{\Theta}^{\varepsilon}\right), t\right), \qquad \mathfrak{Mu}^{\varepsilon}\left(0, \mu, \theta\right) = \mu, \tag{1.20}$$

$$\frac{\partial \mathbf{\Theta}^{\varepsilon}}{\partial t} = -\frac{1}{\varepsilon} + \frac{\cos\left(\mathbf{\Theta}^{\varepsilon}\right)}{\sqrt{2\mathfrak{Mu}^{\varepsilon}}} E_{\varepsilon}\left(\sqrt{2\mathfrak{Mu}^{\varepsilon}}\cos\left(\mathbf{\Theta}^{\varepsilon}\right), t\right), \qquad \mathbf{\Theta}^{\varepsilon}\left(0, \mu, \theta\right) = \theta, \tag{1.21}$$

and we observe that $\mathfrak{Mu}^\varepsilon$ is no longer an invariant.

This kind of situation is very similar to the situation encountered in the Geometrical Gyrokinetic theory that was introduced by Littlejohn [9, 10, 11], Brizard [1], Dubin *et al.* [2], Frieman & Chen [6], Hahm [7], Hahm, Lee & Brizard [8], Parra & Catto [15, 16, 17] and Quin *et al* [18]. In order to study this kind of situation, the idea is to make an infinitesimal change of coordinate $(\mu,\theta) \mapsto \left(\tilde{\mu},\tilde{\theta}\right) = \boldsymbol{\mathcal{L}}_\varepsilon^t(\mu,\theta)$ bringing the characteristics independent of $\tilde{\theta}$ and in which the characteristic associated with $\tilde{\mu}$ is an invariant.

The infinitesimal change of coordinates that we will construct belongs to the class of the Lie change of coordinates that are defined as follow:

Definition 1.1. *A Lie Change of Coordinates is a formal change of coordinates of the form*

$$\boldsymbol{\mathcal{L}}_\varepsilon:\ (\mu,\theta,t) \mapsto \boldsymbol{\mathcal{L}}_\varepsilon(\mu,\theta,t) = \ldots \circ \bar{\varphi}^n_{\varepsilon^n} \circ \ldots \circ \bar{\varphi}^1_\varepsilon(\mu,\theta,t) \tag{1.22}$$
$$= \left(\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{L}}_\varepsilon(\mu,\theta,t), t\right) \tag{1.23}$$

where for each $n \in \mathbb{N}^\star$, $\bar{\varphi}^n_\lambda$ is the flow of a vector field

$$\bar{\mathbf{Z}}^n = \bar{Z}^n_1\partial_\mu + \bar{Z}^n_2\partial_\theta, \tag{1.24}$$

i.e., the solution of

$$\frac{\partial\bar{\varphi}^{n,1}_\lambda}{\partial\lambda} = \bar{Z}^n_1(\bar{\varphi}^n_\lambda), \tag{1.25}$$
$$\frac{\partial\bar{\varphi}^{n,2}_\lambda}{\partial\lambda} = \bar{Z}^n_2(\bar{\varphi}^n_\lambda), \qquad \bar{\varphi}_0(\mu,\theta,t) = (\mu,\theta,t), \tag{1.26}$$
$$\frac{\partial\bar{\varphi}^{n,3}_\lambda}{\partial\lambda} = 0. \tag{1.27}$$

In this paper we will always denote by $\boldsymbol{\mathcal{P}}\varphi = (\varphi_1,\varphi_2)$ the projection of a function $\varphi = (\varphi_1,\varphi_2,\varphi_3)$. In section 3, starting from the Hilbert expansions of the electric field E_ε and the electric potential ϕ_ε

$$E_\varepsilon = E_0 + \varepsilon E_1 + \varepsilon^2 E_2 + \ldots, \tag{1.28}$$
$$\phi_\varepsilon = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \ldots, \tag{1.29}$$

we will develop and use a Lie Transform algorithm, based on the utilization of the Poincaré-Cartan one form, in order to give a constructive proof of the following Theorem:

Theorem 1.1. *There exists a Lie change of coordinates $\boldsymbol{\mathcal{L}}_\varepsilon$ such that in the yielding $\left(\tilde{\mu},\tilde{\theta}\right)$ coordinate system, given by $(r,v_r) \mapsto \left(\tilde{\mu},\tilde{\theta}\right) = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{L}}_\varepsilon\left(\mathfrak{Pol}(r,v_r), t\right)$ where*

$$\mathfrak{Pol}:\ \mathbb{R}^2 \to \mathbb{R}_+\times]-\pi,\pi]\ ;\ (r,v_r) \mapsto (\mu,\theta) \tag{1.30}$$

with θ and μ given by formulas (1.15)-(1.16), *the system of equations* (1.1)-(1.5) *reads:*

$$\frac{\partial \tilde{f}_\varepsilon}{\partial t}\left(\tilde{\mu},\tilde{\theta},t\right)+a_\varepsilon\left(\tilde{\mu},t\right)\frac{\partial \tilde{f}_\varepsilon}{\partial \tilde{\theta}}\left(\tilde{\mu},\tilde{\theta},t\right)=0, \tag{1.31}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi^\varepsilon}{\partial r}\right)=\rho_\varepsilon\left(t,r\right), \quad E_\varepsilon=-\frac{\partial \phi_\varepsilon}{\partial r}, \tag{1.32}$$

$$\rho_\varepsilon\left(t,r\right)=\int_{\mathbf{D}_\varepsilon^t} h_r\left(\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{L}}_\varepsilon^{-1}\left(\tilde{\mu}',\tilde{\theta}',t\right)\right)\tilde{f}_\varepsilon\left(\tilde{\mu}',\tilde{\theta}',t\right)\left|\mathcal{J}_{\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{L}}_\varepsilon^{-1}}\left(\tilde{\mu}',\tilde{\theta}',t\right)\right|d\tilde{\mu}'d\tilde{\theta}', \tag{1.33}$$

$$E_\varepsilon\left(t,r=0\right)=0, \quad \phi_\varepsilon\left(t,r=0\right)=0, \tag{1.34}$$

$$\tilde{f}_\varepsilon\left(\tilde{\mu},\tilde{\theta},t=0\right)=f_0\left(\mathfrak{Pol}^{-1}\circ\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{L}}_\varepsilon^{-1}\left(\tilde{\mu},\tilde{\theta},t=0\right)\right), \tag{1.35}$$

where $\tilde{f}_\varepsilon$ is the particle density expressed in the $\left(\tilde{\mu},\tilde{\theta}\right)$ coordinate system, a_ε is defined by (3.90), *$h_r=h_r\left(\mu',\theta'\right)$ is given by $h_r\left(\mu',\theta'\right)=\delta\left(r-\sqrt{2\mu'}\cos\left(\theta'\right)\right)$, $\left|\mathcal{J}_{\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{L}}_\varepsilon^{-1}}\left(\tilde{\mu}',\tilde{\theta}',t\right)\right|$ is the jacobian associated with $\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{L}}_\varepsilon^{-1}$ and $\mathbf{D}_\varepsilon^t=\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{L}}_\varepsilon\left(\mathbb{R}_+\times]-\pi,\pi],t\right)$.*

Moreover, up to the second order, $\boldsymbol{\mathcal{L}}_\varepsilon$, $\boldsymbol{\mathcal{L}}_\varepsilon^{-1}$ and a_ε admit the following expansions:

$$\begin{aligned}\tilde{\mu}&=\mu+\varepsilon\bar{Z}_1^1\left(\mu,\theta,t\right)+\mathcal{O}\left(\varepsilon^2\right),\\ \tilde{\theta}&=\theta+\varepsilon\bar{Z}_2^1\left(\mu,\theta,t\right)+\mathcal{O}\left(\varepsilon^2\right),\end{aligned} \tag{1.36}$$

$$\begin{aligned}\mu&=\tilde{\mu}-\varepsilon\bar{Z}_1^1\left(\tilde{\mu},\tilde{\theta},t\right)+\mathcal{O}\left(\varepsilon^2\right),\\ \theta&=\tilde{\theta}-\varepsilon\bar{Z}_2^1\left(\tilde{\mu},\tilde{\theta},t\right)+\mathcal{O}\left(\varepsilon^2\right),\end{aligned} \tag{1.37}$$

and

$$a_\varepsilon\left(\tilde{\mu},t\right)=-\frac{1}{\varepsilon}+\frac{1}{2\pi\sqrt{2\tilde{\mu}}}\int_{-\pi}^{\pi}\cos\left(\tilde{\theta}\right)E_0\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right)d\tilde{\theta}+\mathcal{O}\left(\varepsilon\right), \tag{1.38}$$

where $\bar{Z}_1^1$ and $\bar{Z}_2^1$ are given by formula (3.41) *and* (3.49).

Remark 1.1. *In formulas* (1.36), (1.37) *and* (1.38), *we have only given the second order expansions of the direct and the reciprocal Lie change of coordinates and the first order expansion of a_ε. Nevertheless the algorithm developed in the proof of Theorem 1.1 allows us to obtain these expansions at any order.*

The change of coordinates $\boldsymbol{\mathcal{L}}_\varepsilon$ is formal in the sense that $\boldsymbol{\mathcal{L}}_\varepsilon$ corresponds to a composition of an infinite number of flows. Moreover the construction of $\boldsymbol{\mathcal{L}}_\varepsilon$ is based on Lie series expansions of each of these flows; i.e., for any $n\in\mathbb{N}$ we will use the formal expansion

$$\bar{\varphi}_{\varepsilon^n}^n=\sum_{n\geq 0}\frac{\varepsilon^n}{n!}\mathbf{Z}^n\cdot.$$

See [14] (page 31) for more precisions about these series.

Making first order approximations in the characteristics and in the change of coordinates, we will use (1.31)-(1.35) in order to simulate the solution f_ε of (1.1)-(1.5). More precisely, approximating the change of coordinates by

$$\tilde{\mu} = \mu + \mathcal{O}(\varepsilon), \tag{1.39}$$

$$\tilde{\theta} = \theta + \mathcal{O}(\varepsilon), \tag{1.40}$$

the electric field and the electric potential by

$$E_\varepsilon = E_0 + \mathcal{O}(\varepsilon), \tag{1.41}$$

$$\phi_\varepsilon = \phi_0 + \mathcal{O}(\varepsilon), \tag{1.42}$$

the charge density as follow:

$$\begin{aligned}
&\mathbf{D}_\varepsilon^t = \boldsymbol{\mathcal{PL}}_\varepsilon(\mathbb{R}_+\times]-\pi,\pi],t) \simeq \boldsymbol{\mathcal{PL}}_0(\mathbb{R}_+\times]-\pi,\pi],t) = \mathbb{R}_+\times]-\pi,\pi],\\
&h_r\left(\boldsymbol{\mathcal{PL}}_\varepsilon^{-1}\left(\tilde{\mu}',\tilde{\theta}',t\right)\right) \simeq h_r\left(\boldsymbol{\mathcal{PL}}_0^{-1}\left(\tilde{\mu}',\tilde{\theta}',t\right)\right) = h_r\left(\tilde{\mu}',\tilde{\theta}'\right),\\
&\left|\mathcal{J}_{\boldsymbol{\mathcal{PL}}_\varepsilon^{-1}}\left(\tilde{\mu}',\tilde{\theta}',t\right)\right| \simeq \left|\mathcal{J}_{\boldsymbol{\mathcal{PL}}_0^{-1}}\left(\tilde{\mu}',\tilde{\theta}',t\right)\right| = 1,\\
&\rho_\varepsilon(t,r) \simeq \int_{\mathbb{R}_+\times]-\pi,\pi]} h_r\left(\tilde{\mu}',\tilde{\theta}'\right)\tilde{f}_\varepsilon\left(\tilde{\mu}',\tilde{\theta}',t'\right)d\tilde{\mu}'d\tilde{\theta}',
\end{aligned} \tag{1.43}$$

that is

$$\rho_\varepsilon(t,r) \simeq \int_{\mathbb{R}_+\times]-\pi,\pi]} \delta\left(r-\sqrt{2\tilde{\mu}'}\cos\left(\tilde{\theta}'\right)\right)\tilde{f}_\varepsilon\left(\tilde{\mu}',\tilde{\theta}',t\right)d\tilde{\mu}'d\tilde{\theta}'. \tag{1.44}$$

and a_ε by

$$a_\varepsilon(\tilde{\mu},t) \simeq -\frac{1}{\varepsilon} + \frac{1}{2\pi\sqrt{2\tilde{\mu}}}\int_{-\pi}^{\pi}\cos\left(\tilde{\theta}\right)E_0\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right)d\tilde{\theta}, \tag{1.45}$$

we obtain:

$$\frac{\partial\tilde{f}_\varepsilon}{\partial t} + \left(-\frac{1}{\varepsilon} + \frac{1}{2\pi\sqrt{2\tilde{\mu}}}\int_{-\pi}^{\pi}\cos\left(\tilde{\theta}\right)E_\varepsilon\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right)d\tilde{\theta}\right)\frac{\partial\tilde{f}_\varepsilon}{\partial\tilde{\theta}} = 0, \tag{1.46}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi_\varepsilon}{\partial r}\right) = \int_{\mathbb{R}_+\times]-\pi,\pi]}\delta\left(r-\sqrt{2\tilde{\mu}'}\cos\left(\tilde{\theta}'\right)\right)\tilde{f}_\varepsilon\left(\tilde{\mu}',\tilde{\theta}',t\right)d\tilde{\mu}'d\tilde{\theta}', \tag{1.47}$$

$$E_\varepsilon = -\frac{\partial\phi_\varepsilon}{\partial r}, \tag{1.48}$$

$$E_\varepsilon(t,r=0) = 0, \quad \phi_\varepsilon(t,r=0) = 0, \tag{1.49}$$

$$\tilde{f}_\varepsilon\left(\tilde{\mu},\tilde{\theta},t=0\right) = f_0\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),\sqrt{2\tilde{\mu}}\sin\left(\tilde{\theta}\right)\right). \tag{1.50}$$

We will give some remarks about this approximation in Subsection 3.5.

In the last section we will simulate (1.46)-(1.50) and then we will obtain an approximation of f_ε through:

$$f_\varepsilon(r,v_r,t) \simeq \tilde{f}_\varepsilon(\mu,\theta,t). \tag{1.51}$$

The numerical method that we will use to simulate (1.46)-(1.50) will be a Particle in Cell (PIC) method. I recall that a PIC method consists in the coupling of a particle method for Vlasov, and a mesh method for Poisson. The principle of the method is to discretize the distribution function by a set of macro-particles and to advance them in time by numerically solving the dynamical system giving the characteristics. As a consequence, solving this dynamical system in the new system of coordinates, reduces to find a trajectory in $\mathbb{R}$, in place of a trajectory in $\mathbb{R}^2$ when it is solved in the original system of coordinates.

The paper is organized as follows: in Section 2 we will construct an odd dimensional differential manifold well adapted to the study of (1.6)-(1.7) and we will give the mathematical tools necessary for the comprehension of the Lie Transform method we develop then. As a by product of this section we obtain that the non autonomous dynamical system we work with is characterized intrinsically by an autonomous dynamical system on the odd differential manifold we work within. Moreover, we will see that this autonomous dynamical system can also be characterized by the equivalence class of a differential one form called the Poincaré Cartan one form. Furthermore, we will introduce the Noether Theorem within this framework. This Theorem gives essentially an intuitive help for the comprehension of the Lie Transform method. In the third section, we will set out the Lie transform method and we will use it in order to derive the Lie Coordinate System and to prove Theorem 1.1. Finally, in the fourth and fifth section, we will implement and test the previously described numerical method based on the Lie transform method analysis.

2 Geometrical Tools

2.1 Characterization of the differential system (1.6)-(1.7) and of the Vlasov equation on an odd dimensional manifold

In the present subsection we will characterize intrinsically on an odd dimensional manifold differential systems of the form

$$\frac{\partial \mathbf{R}_G^\varepsilon}{\partial t} = \partial_{v_r} G_\varepsilon \left(\mathbf{R}_G^\varepsilon, \mathbf{V}_{r,G}^\varepsilon, t\right), \qquad \mathbf{R}_G^\varepsilon\left(0, r, v_r\right) = r, \tag{2.1}$$

$$\frac{\partial \mathbf{V}_{r,G}^\varepsilon}{\partial t} = -\partial_r G_\varepsilon \left(\mathbf{R}_G^\varepsilon, \mathbf{V}_{r,G}^\varepsilon, t\right), \qquad \mathbf{V}_{r,G}^\varepsilon\left(0, r, v_r\right) = v_r, \tag{2.2}$$

where $G_\varepsilon = G_\varepsilon(r, v_r, t)$ is a smooth function, and PDEs

$$\frac{\partial f_\varepsilon^G}{\partial t}\left(r, v_r, t\right) + \partial_{v_r} G_\varepsilon\left(r, v_r, t\right) \frac{\partial f_\varepsilon^G}{\partial r}\left(r, v_r, t\right) - \partial_r G_\varepsilon\left(r, v_r, t\right) \frac{\partial f_\varepsilon^G}{\partial v_r}\left(r, v_r, t\right) = 0 \tag{2.3}$$

of unknown f_ε^G, through a vector field $\boldsymbol{\tau}_G^\varepsilon$. Notice that if $G_\varepsilon = H_\varepsilon$, where H_ε is given by formula (1.8), dynamical system (2.1)-(2.2) and PDE (2.3) coincide with dynamical system (1.9)-(1.10) and PDE (1.1). The principal results are given in theorem 2.1 and 2.2.

Firstly, we need to build the manifold on which we will work. As a topological space we take $\mathcal{M} = \mathbb{R}^2 \times \mathbb{R}_+$ endowed with the (r, v_r, t) coordinate system and with its usual topology. Concerning the differential structure, we choose the differential atlas $\mathcal{A}$ which contains all the coordinate charts of type $(\mathcal{U}, \varphi)$, where $\varphi :\ \mathcal{U} \to \mathbb{R}^3;\ (r, v_r, t) \mapsto (\boldsymbol{\mathcal{P}}\varphi\left(r, v_r, t\right), t)$, which are compatible with the global coordinate chart $(\mathcal{M}, \mathfrak{G})$, where $\mathfrak{G} : \mathcal{M} \to \mathbb{R}^3; (r, v_r, t) \mapsto \mathfrak{G}(r, v_r, t) = (r, v_r, t)$, and which leave the last coordinate t unchanged.

Defining the vector field $\mathbf{X}_G^\varepsilon$ by:

$$\mathbf{X}_G^\varepsilon = \partial_{v_r} G_\varepsilon \partial_r - \partial_r G_\varepsilon \partial_{v_r} + \partial_t, \tag{2.4}$$

and denoting by $\mathbf{F}_{\lambda,G}^\varepsilon$ its flow; i.e., the solution of

$$\frac{\partial \mathbf{F}_{\lambda,G}^{\varepsilon,1}}{\partial \lambda} = \partial_{v_r} G_\varepsilon \left(\mathbf{F}_{\lambda,G}^\varepsilon\right), \qquad \mathbf{F}_{0,G}^{\varepsilon,1}(r, v_r, t) = r, \tag{2.5}$$

$$\frac{\partial \mathbf{F}_{\lambda,G}^{\varepsilon,2}}{\partial \lambda} = -\partial_r G_\varepsilon \left(\mathbf{F}_{\lambda,G}^\varepsilon\right), \qquad \mathbf{F}_{0,G}^{\varepsilon,2}(r, v_r, t) = v_r, \tag{2.6}$$

$$\frac{\partial \mathbf{F}_{\lambda,G}^{\varepsilon,3}}{\partial \lambda} = 1, \qquad \mathbf{F}_{0,G}^{\varepsilon,3}(r, v_r, t) = t, \tag{2.7}$$

we conclude that the trajectory associated with (2.1)-(2.2) corresponds to

$$\left(\mathbf{F}_{t,G}^{1,\varepsilon}(r, v_r, 0), \mathbf{F}_{t,G}^{2,\varepsilon}(r, v_r, 0)\right). \tag{2.8}$$

Now, we have enough material to characterize intrinsically the solution of (2.1)-(2.2).

Theorem 2.1. *Let $\boldsymbol{\tau}_G^\varepsilon : \mathcal{M} \to T\mathcal{M}$ be the vector field whose principal part in the (r, v_r, t) coordinate system is given by $\mathbf{X}_G^\varepsilon$, defined by formula* (2.4)*, and let $\boldsymbol{\mathcal{F}}_{\lambda,G}^\varepsilon$ be its flow. Then, in every coordinate system $(\tilde{r}, \tilde{v}_r, t)$ belonging to $\mathcal{A}$ the trajectory associated with the dynamical system* (1.6)-(1.7) *is given by $\left(\tilde{\mathbf{F}}_{t,G}^{1,\varepsilon}(\tilde{r}, \tilde{v_r}, 0), \tilde{\mathbf{F}}_{t,G}^{2,\varepsilon}(\tilde{r}, \tilde{v_r}, 0)\right)$, where $\tilde{\mathbf{F}}_{\lambda,G}^\varepsilon$ corresponds to the representative of $\boldsymbol{\mathcal{F}}_{\lambda,G}^\varepsilon$ in the $(\tilde{r}, \tilde{v}_r, t)$ coordinate system, or equivalently to the flow of $\tilde{\mathbf{X}}_G^\varepsilon$, where $\tilde{\mathbf{X}}_G^\varepsilon$ corresponds to the representative of the principal part of $\boldsymbol{\tau}_G^\varepsilon$ in the $(\tilde{r}, \tilde{v}_r, t)$ coordinate system.*

Proof. Let $\mathbf{F}_{\lambda,G}^\varepsilon$ be the flow of $\mathbf{X}_G^\varepsilon$, where $\mathbf{X}_G^\varepsilon$ is given by (2.4). We denote by $\mathbf{R}^\star \equiv \mathbf{R}^\star(\lambda, r, v_r, t)$, $\mathbf{V}_r^\star \equiv \mathbf{V}_r^\star(\lambda, r, v_r, t)$ and $\mathbf{T}^\star \equiv \mathbf{T}^\star(\lambda, r, v_r, t)$ its components. Notice that $\mathbf{R}^\star$ and $\mathbf{V}_r^\star$ depends on the small parameter ε. But since this dependency does not play a role in this proof, we do not precise it in the notation. Then, (2.8) reads:

$$\begin{aligned} &\mathbf{R}^\star(t, r, v_r, 0) = \mathbf{R}_G(r, v_r, t), \\ &\mathbf{V}_r^\star(t, r, v_r, 0) = \mathbf{V}_{r,G}(r, v_r, t), \\ &\mathbf{T}^\star(t, r, v_r, 0) = t. \end{aligned} \tag{2.9}$$

Let

$$\psi \ : \ (r, v_r, t) \mapsto \left(\tilde{r}, \tilde{v}_r, \tilde{t}\right) = \left(\boldsymbol{\mathcal{P}}\psi(r, v_r, t), t\right)$$

be a change of coordinates such that $\tilde{t} = t$. We denote by $\tilde{\mathbf{R}}^\star \equiv \tilde{\mathbf{R}}^\star\left(\lambda, \tilde{r}, \tilde{v}_r, \tilde{t}\right)$, $\tilde{\mathbf{V}}_r^\star \equiv \tilde{\mathbf{V}}_{\mathbf{r}}^\star\left(\lambda, \tilde{r}, \tilde{v}_r, \tilde{t}\right)$ and $\tilde{\mathbf{T}}^\star \equiv \tilde{\mathbf{T}}^\star\left(\lambda, \tilde{r}, \tilde{v}_r, \tilde{t}\right)$ the components of $\tilde{\mathbf{F}}_{\lambda,G}^\varepsilon$; i.e., the components of the expression of the flow in the $\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right)$ coordinate system. Then, the usual change of coordinates rules yield:

$$
\begin{aligned}
&\tilde{\mathbf{R}}^{\star}\left(\lambda, \tilde{r}, \tilde{v}_r, \tilde{t}\right) = \psi_1\left(\mathbf{R}^{\star}\left(\lambda, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right), \tilde{t}\right), \mathbf{V}_r^{\star}\left(\lambda, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right), \tilde{t}\right), \mathbf{T}^{\star}\left(\lambda, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right), \tilde{t}\right)\right),\\
&\tilde{\mathbf{V}}_{\mathbf{r}}^{\star}\left(\lambda, \tilde{r}, \tilde{v}_r, \tilde{t}\right) = \psi_2\left(\mathbf{R}^{\star}\left(\lambda, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right), \tilde{t}\right), \mathbf{V}_r^{\star}\left(\lambda, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right), \tilde{t}\right), \mathbf{T}^{\star}\left(\lambda, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right), \tilde{t}\right)\right),\\
&\tilde{\mathbf{T}}^{\star}\left(\lambda, \tilde{r}, \tilde{v}_r, \tilde{t}\right) = \mathbf{T}^{\star}\left(\lambda, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right), \tilde{t}\right).
\end{aligned}
\tag{2.10}
$$

On the other hand, let $\tilde{\mathbf{R}}_G \equiv \tilde{\mathbf{R}}_G\left(\tilde{r}, \tilde{v}_r, t\right)$ and $\tilde{\mathbf{V}}_{\mathbf{r},\mathbf{G}} \equiv \tilde{\mathbf{V}}_{\mathbf{r},\mathbf{G}}\left(\tilde{r}, \tilde{v}_r, t\right)$ be the components of the trajectory whose range by $\boldsymbol{\mathcal{P}}\psi^{-1}$ is the trajectory associate with $\mathbf{R}_G\left(r, v_r, t\right)$ and $\mathbf{V}_{r,G}\left(r, v_r, t\right)$; i.e., such that

$$
\begin{aligned}
&\left(\tilde{\mathbf{R}}_G\left(\tilde{r}, \tilde{v}_r, t\right), \tilde{\mathbf{V}}_{r,G}\left(\tilde{r}, \tilde{v}_r, t\right)\right)\\
=&(\psi_1\left(\mathbf{R}_G\left(\boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), t\right), \mathbf{V}_{r,G}\left(\boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), t\right), t\right),\\
&\psi_2\left(\mathbf{R}_G\left(\boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), t\right), \mathbf{V}_{r,G}\left(\boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), t\right), t\right)).
\end{aligned}
$$

To finish the proof, we have to show that

$$
\left(\tilde{\mathbf{R}}^{\star}\left(t, \tilde{r}, \tilde{v}_r, 0\right), \tilde{\mathbf{V}}_r^{\star}\left(t, \tilde{r}, \tilde{v}_r, 0\right)\right) = \left(\tilde{\mathbf{R}}_G\left(\tilde{r}, \tilde{v}_r, t\right), \tilde{\mathbf{V}}_{r,G}\left(\tilde{r}, \tilde{v}_r, t\right)\right). \tag{2.11}
$$

Differentiating

$$
\tilde{\mathbf{T}}^{\star}\left(\lambda, \tilde{r}, \tilde{v}_r, \tilde{t}\right) = \mathbf{T}^{\star}\left(\lambda, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right), \tilde{t}\right)
$$

with respect to λ yields:

$$
\frac{\partial \tilde{\mathbf{T}}^{\star}}{\partial \lambda} = 1
$$

and consequently $\tilde{\mathbf{T}}^{\star}\left(\tilde{t}, r, v_r, 0\right) = \tilde{t}$. Hence, we obtain:

$$
\begin{aligned}
&\left(\tilde{\mathbf{R}}^{\star}\left(\tilde{t}, \tilde{r}, \tilde{v}_r, 0\right), \tilde{\mathbf{V}}_{\mathbf{r}}^{\star}\left(\tilde{t}, \tilde{r}, \tilde{v}_r, 0\right)\right)\\
=&(\psi_1\left(\mathbf{R}^{\star}\left(\tilde{t}, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), 0\right), \mathbf{V}_r^{\star}\left(\tilde{t}, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), 0\right), \tilde{t}\right),\\
&\psi_2\left(\mathbf{R}^{\star}\left(\tilde{t}, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), 0\right), \mathbf{V}_r^{\star}\left(\tilde{t}, \boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), 0\right), \tilde{t}\right)).
\end{aligned}
\tag{2.12}
$$

Finally, using (2.9) we obtain:

$$
\begin{aligned}
&\left(\tilde{\mathbf{R}}^{\star}\left(\tilde{t}, \tilde{r}, \tilde{v}_r, 0\right), \tilde{\mathbf{V}}_{\mathbf{r}}^{\star}\left(\tilde{t}, \tilde{r}, \tilde{v}_r, 0\right)\right)\\
=&(\psi_1\left(\mathbf{R}_G\left(\boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), \tilde{t}\right), \mathbf{V}_{r,G}\left(\boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), \tilde{t}\right), \tilde{t}\right),\\
&\psi_2\left(\mathbf{R}_G\left(\boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), \tilde{t}\right), \mathbf{V}_{r,G}\left(\boldsymbol{\mathcal{P}}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, 0\right), \tilde{t}\right), \tilde{t}\right))\\
=&\left(\tilde{\mathbf{R}}_G\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right), \tilde{\mathbf{V}}_{r,G}\left(\tilde{r}, \tilde{v}_r, \tilde{t}\right)\right)
\end{aligned}
\tag{2.13}
$$

This ends the proof of Theorem 2.1. □

Theorem 2.2. *Let $\boldsymbol{\tau}_G^\varepsilon : \mathcal{M} \to T\mathcal{M}$ be the vector field whose principal part in the (r, v_r, t) coordinate system is given by $\mathbf{X}_G^\varepsilon$, defined by formula (2.4). Then, in every coordinate system $(\tilde{r}, \tilde{v}_r, t)$ belonging to $\mathcal{A}$ the PDE (2.3) is given by*

$$i_{\tilde{\mathbf{X}}_G^\varepsilon} d\tilde{f}_\varepsilon^G = 0, \tag{2.14}$$

where $\tilde{\mathbf{X}}_G^\varepsilon$ and $\tilde{f}_\varepsilon^G$ correspond respectively to the representative of the principal part of $\boldsymbol{\tau}_G^\varepsilon$ and the representative of f_ε^G in the $(\tilde{r}, \tilde{v}_r, t)$ coordinate system.

Proof. Firstly in the (r, v_r, t) coordinate system $i_{\mathbf{X}_G^\varepsilon} df_\varepsilon^G$ reads:

$$\begin{aligned}
i_{\mathbf{X}_G^\varepsilon} df_\varepsilon^G &= \left(\nabla_{(r,v_r,t)} f_\varepsilon^G\right)^T \mathbf{X}_G^\varepsilon \\
&= \frac{\partial f_\varepsilon^G}{\partial t} + X_G^{\varepsilon,1} \frac{\partial f_\varepsilon^G}{\partial r} + X_G^{\varepsilon,2} \frac{\partial f_\varepsilon^G}{\partial v_r} \\
&= \frac{\partial f_\varepsilon^G}{\partial t} + \partial_{v_r} G_\varepsilon \frac{\partial f_\varepsilon^G}{\partial r} - \partial_r G_\varepsilon \frac{\partial f_\varepsilon^G}{\partial v_r} \\
&= 0,
\end{aligned}$$

and (2.14) is satisfied. Now, let $(\tilde{r}, \tilde{v}_r, t)$ be a coordinate system belonging to $\mathcal{A}$ and $(\mathcal{U}, \psi) \in \boldsymbol{\mathcal{A}}$ the corresponding coordinate chart. Then, the expression of $\boldsymbol{\tau}_G^\varepsilon$ is given by:

$$\tilde{\mathbf{X}}_G^\varepsilon (\tilde{r}, \tilde{v}_r, t) = \nabla_{(r,v_r,t)} \psi \left(\psi^{-1} (\tilde{r}, \tilde{v}_r, t)\right) \mathbf{X}_G^\varepsilon \left(\psi^{-1} (\tilde{r}, \tilde{v}_r, t)\right), \tag{2.15}$$

and the expression of the particle distribution is given by:

$$\tilde{f}_\varepsilon^G (\tilde{r}, \tilde{v}_r, t) = f_\varepsilon^G \left(\psi^{-1} (\tilde{r}, \tilde{v}_r, t)\right). \tag{2.16}$$

Consequently $i_{\tilde{\mathbf{X}}_G^\varepsilon} d\tilde{f}_\varepsilon^G$ reads:

$$\begin{aligned}
& i_{\tilde{\mathbf{X}}_G^\varepsilon} d\tilde{f}_\varepsilon^G \\
&= \left(\nabla_{(\tilde{r},\tilde{v}_r,t)} \tilde{f}_\varepsilon^G\right)^T \tilde{\mathbf{X}}_G^\varepsilon \\
&= \left(\left(\nabla_{(r,v_r,t)} \psi \left(\psi^{-1} (\tilde{r}, \tilde{v}_r, t)\right)\right)^{-T} \nabla_{(\tilde{r},\tilde{v}_r,t)} f_\varepsilon^G \left(\psi^{-1} (\tilde{r}, \tilde{v}_r, t)\right)\right)^T \left(\nabla_{(r,v_r,t)} \psi \left(\psi^{-1} (\tilde{r}, \tilde{v}_r, t)\right) \mathbf{X}_G^\varepsilon \left(\psi^{-1} (\tilde{r}, \tilde{v}_r, t)\right)\right) \\
&= \left(\nabla_{(\tilde{r},\tilde{v}_r,t)} f_\varepsilon^G \left(\psi^{-1} (\tilde{r}, \tilde{v}_r, t)\right)\right)^T \mathbf{X}_G^\varepsilon \left(\psi^{-1} (\tilde{r}, \tilde{v}_r, t)\right) \\
&=0,
\end{aligned} \tag{2.17}$$

and (2.14) is satisfied.

□

Since the last coordinates of $\tilde{\mathbf{X}}_G^\varepsilon$ is always equal to 1, equation (2.14) reads also:

$$\frac{\partial \tilde{f}_\varepsilon^G}{\partial t} + \tilde{X}_G^{\varepsilon,1} \frac{\partial \tilde{f}_\varepsilon^G}{\partial \tilde{r}} + \tilde{X}_G^{\varepsilon,2} \frac{\partial \tilde{f}_\varepsilon^G}{\partial \tilde{v}_r} = 0. \tag{2.18}$$

2.2 The Poincaré Cartan one-form

Theorems 2.1 and 2.2 allow us to characterize intrinsically the differential system (2.1)-(2.2) and the PDE (2.3). More precisely, these Theorems ensure us that the differential system (2.1)-(2.2) and the PDE (2.3) are characterized intrinsically through the vector field $\boldsymbol{\tau}_G^\varepsilon$. Now, we will see that $\boldsymbol{\tau}_G^\varepsilon$ can also be characterized by an equation that involves a differential one form $\boldsymbol{\gamma}_G^\varepsilon$ called the Poincaré-Cartan one-form. We will essentially see that $\boldsymbol{\tau}_G^\varepsilon$ can be characterized as the direction vector of the eigenspace of $d\boldsymbol{\gamma}_G^\varepsilon$ associated with the eigenvalue 0 and whose last component is 1. In other words we will see that $\boldsymbol{\tau}_G^\varepsilon$ is the unique solution of $i_{\boldsymbol{\tau}_G^\varepsilon} d\boldsymbol{\gamma}_G^\varepsilon = 0$ satisfying $\boldsymbol{\tau}_{G,3}^\varepsilon = 1$. Afterwards, we will introduce the following equivalence relation on the one forms space : "$\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ if and only if $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is exact", and we will see that $\forall \boldsymbol{\beta}_G^\varepsilon \in [\boldsymbol{\gamma}_G^\varepsilon]$, where $[\boldsymbol{\gamma}_G^\varepsilon]$ stands for the equivalence class of $\boldsymbol{\gamma}_G^\varepsilon$, the vector field $\boldsymbol{\tau}_G^\varepsilon$ is characterized by $i_{\boldsymbol{\tau}_G^\varepsilon} d\boldsymbol{\beta}_G^\varepsilon = 0$ and $\boldsymbol{\tau}_{G,3}^\varepsilon = 1$. The main results are summarized in theorem 2.3.

Definition 2.1. *The Poincaré-Cartan 1-form $\boldsymbol{\gamma}_G^\varepsilon$ associated with the dynamical system* (2.1)-(2.2) *is the one-form whose expression in the (r, v_r, t) coordinate system is given by:*

$$\boldsymbol{\Gamma}_G^\varepsilon (r, v_r, t) = v_r dr - G_\varepsilon dt. \tag{2.19}$$

The matrix associated with the differential two-form $d\boldsymbol{\Gamma}_G^\varepsilon$ is given by

$$M_G^\varepsilon (r, v_r, t) = \begin{pmatrix} 0 & -1 & -\partial_r G_\varepsilon \\ 1 & 0 & -\partial_{v_r} G_\varepsilon \\ \partial_r G_\varepsilon & \partial_{v_r} G_\varepsilon & 0 \end{pmatrix} \tag{2.20}$$

Lemma 2.1. *Let $(\tilde{r}, \tilde{v}_r, t)$ be a coordinate system belonging to $\boldsymbol{\mathcal{A}}$ and $\tilde{M}_G^\varepsilon$ the matrix associated with the representative of $d\boldsymbol{\gamma}_G^\varepsilon$ in this coordinate system. Then,*

$$\mathbf{Ker}\left(\tilde{M}_G^\varepsilon (\tilde{r}, \tilde{v}_r, t)\right) = \mathbf{vect}\left(\tilde{\mathbf{X}}_G (\tilde{r}, \tilde{v}_r, t)\right). \tag{2.21}$$

Proof. Let M_G^ε be the matrix defined by (2.20). Since M_G^ε is antisymmetric, its maximal rank is 2. As $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is of rank 2, the rank of M_G^ε is exactly 2. Moreover,

$$i_{\mathbf{X}_G^\varepsilon} d\boldsymbol{\Gamma}_G^\varepsilon (r, v_r, t) = \left(\mathbf{X}_G^\varepsilon (r, v_r, t)\right)^T M_G^\varepsilon (r, v_r, t) = 0. \tag{2.22}$$

Since, $\forall (r, v_r, t)\,,\ \mathbf{X}_G^\varepsilon (r, v_r, t) \neq 0$ (the last component is 1) we have:

$$\mathbf{Ker}\left(M_G^\varepsilon (r, v_r, t)\right) = \mathbf{vect}\left(\mathbf{X}_G^\varepsilon (r, v_r, t)\right). \tag{2.23}$$

Let

$$\psi \ : \ (r, v_r, t) \mapsto \left(\tilde{r}, \tilde{v}_r, \tilde{t}\right) = \left(\boldsymbol{\mathcal{P}}\psi (r, v_r, t), t\right)$$

be a change of coordinates belonging in $\boldsymbol{\mathcal{A}}$ and $d\tilde{\mathbf{\Gamma}}_G^\varepsilon$ be the expression of $d\boldsymbol{\gamma}_G^\varepsilon$ in the $(\tilde{r}, \tilde{v}_r, t)$ coordinate system. Then, the usual change of coordinates rules for differential two-forms yield:

$$\left\langle d\tilde{\mathbf{\Gamma}}_G^\varepsilon\left(\tilde{r}, \tilde{v}_r, t\right); \tilde{\mathbf{u}}, \tilde{\mathbf{v}}\right\rangle = \left\langle d\mathbf{\Gamma}_G^\varepsilon\left(\psi^{-1}\left(\tilde{r}, \tilde{v}_r, t\right), t\right); d\psi^{-1}_{(\tilde{r}, \tilde{v}_r, t)} \cdot \tilde{\mathbf{u}}, d\psi^{-1}_{(\tilde{r}, \tilde{v}_r, t)} \cdot \tilde{\mathbf{v}}\right\rangle, \tag{2.24}$$

and consequently the expression of $\tilde{M}_G^\varepsilon$ is given by

$$\tilde{M}_G^\varepsilon\left(\tilde{r}, \tilde{v}_r, t\right) = \left(\nabla_{(\tilde{r}, \tilde{v}_r, t)}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, t\right)\right)^T M_G^\varepsilon\left(\psi^{-1}\left(\tilde{r}, \tilde{v}_r, t\right)\right) \nabla_{(\tilde{r}, \tilde{v}_r, t)}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, t\right). \tag{2.25}$$

Notice that formula (2.25) implies that $\tilde{M}_G^\varepsilon$ is of rank 2.

On an other hand the usual change of coordinates rule for vector fields yields that the representative of $\boldsymbol{\tau}_G^\varepsilon$ in the $(\tilde{r}, \tilde{v}_r, t)$ coordinate system is given by:

$$\tilde{\mathbf{X}}_G^\varepsilon\left(\tilde{r}, \tilde{v}_r, t\right) = \nabla_{(r, v_r, t)}\psi\left(\psi^{-1}\left(\tilde{r}, \tilde{v}_r, t\right)\right) \mathbf{X}_G^\varepsilon\left(\psi^{-1}\left(\tilde{r}, \tilde{v}_r, t\right)\right). \tag{2.26}$$

Consequently, the last component of $\tilde{\mathbf{X}}_G^\varepsilon$ is 1 and

$$\begin{aligned} i_{\tilde{\mathbf{X}}_G^\varepsilon} d\tilde{\mathbf{\Gamma}}^\varepsilon &= \left(\tilde{\mathbf{X}}_G^\varepsilon\left(\tilde{r}, \tilde{v}_r, t\right)\right)^T \tilde{M}_G^\varepsilon\left(\tilde{r}, \tilde{v}_r, t\right) \\ &= \left(\mathbf{X}_G^\varepsilon\left(\psi^{-1}\left(\tilde{r}, \tilde{v}_r, t\right)\right)\right)^T M_G^\varepsilon\left(\psi^{-1}\left(\tilde{r}, \tilde{v}_r, t\right)\right) \nabla_{(\tilde{r}, \tilde{v}_r, t)}\psi^{-1}\left(\tilde{r}, \tilde{v}_r, t\right) \\ &= 0. \end{aligned} \tag{2.27}$$

Hence,

$$\mathbf{Ker}\left(\tilde{M}_G^\varepsilon\left(\tilde{r}, \tilde{v}_r, t\right)\right) = \mathbf{vect}\left(\tilde{\mathbf{X}}_G^\varepsilon\left(\tilde{r}, \tilde{v}_r, t\right)\right). \tag{2.28}$$

This ends the proof of Lemma 2.1. □

In particular, lemma 2.1 implies that in every coordinate system the dimension of the kernel of $\tilde{M}_G^\varepsilon$ is equal to 1. Now, these kernels can be characterize intrinsically on the manifold as follow:

Definition 2.2. *The subspace $\mathcal{V}_{(r, v_r, t)} = \left\{c\boldsymbol{\xi}_{(r, v_r, t)}/c \in \mathbb{R}\right\} \subset T_{(r, v_r, t)}\mathcal{M}$, where $\boldsymbol{\xi}_{(r, v_r, t)} \in T_{(r, v_r, t)}\mathcal{M}$ is a vector satisfying $\boldsymbol{\xi}_{(r, v_r, t)} \neq 0$ and*

$$i_{\boldsymbol{\xi}_{(r, v_r, t)}}\left(d\boldsymbol{\gamma}_G^\varepsilon\right)\left(r, v_r, t\right) = 0, \tag{2.29}$$

is called the vortex line of $\boldsymbol{\gamma}_G^\varepsilon$ at (r, v_r, t).

Easy computations lead that the vortex line is well defined; i.e., compatible with the differential structure. Moreover, Lemma 2.1 means that $\forall\,(r, v_r, t) \in \mathcal{M}$, $\boldsymbol{\tau}_G^\varepsilon\left(r, v_r, t\right)$ is the unique generator of $\mathcal{V}_{(r, v_r, t)}$ whose last component is 1.

Proposition 2.1. *Let $(\tilde{r}, \tilde{v}_r, t)$ be local coordinates on $\mathcal{M}$ and let $\tilde{\boldsymbol{X}}_G^\varepsilon$ be the representative of $\boldsymbol{\tau}_G^\varepsilon$ in this coordinates system. Then, $\tilde{\boldsymbol{X}}_G^\varepsilon$ is the unique solution of the equation of unknown $\tilde{\mathbf{Y}}^\varepsilon$*

$$i_{\tilde{\mathbf{Y}}^\varepsilon} d\tilde{\mathbf{\Gamma}}_G^\varepsilon = 0 \tag{2.30}$$

that satisfies $\tilde{\mathbf{Y}}_3^\varepsilon = 1$.

Proposition 2.1 allows us to characterize intrinsically $\boldsymbol{\tau}_G^\varepsilon$ by using $\boldsymbol{\gamma}_G^\varepsilon$. In fact, as $d \circ d = 0$, replacing in (2.29) $\boldsymbol{\gamma}_G^\varepsilon$ by $\boldsymbol{\gamma}_G^\varepsilon + dS_\varepsilon$, where S_ε is a smooth function, yields the same result. As a consequence, we will introduce the following equivalence relation:

Definition 2.3. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two differential one forms. We say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are equivalent if there exists a smooth function S such that $\boldsymbol{\alpha} - \boldsymbol{\beta} = dS$. We will denote by $[\boldsymbol{\alpha}]$ the equivalence class of $\boldsymbol{\alpha}$.*

Then we can generalize Proposition 2.1.

Theorem 2.3. *Let $(\tilde{r}, \tilde{v}_r, t)$ be local coordinates on $\mathcal{M}$, $\tilde{\boldsymbol{X}}_G^\varepsilon$ the representative of $\boldsymbol{\tau}_G^\varepsilon$ in this coordinate system, and $\boldsymbol{\beta}_G^\varepsilon \in [\boldsymbol{\gamma}_G^\varepsilon]$. Then, $\tilde{\boldsymbol{X}}_G^\varepsilon$ is the unique solution of the equation of unknown $\tilde{\mathbf{Y}}^\varepsilon$:*

$$i_{\tilde{\mathbf{Y}}^\varepsilon} d\tilde{\boldsymbol{\beta}}_G^\varepsilon = 0, \tag{2.31}$$

that satisfies $\tilde{\mathbf{Y}}_3^\varepsilon = 1$, where $\tilde{\boldsymbol{\beta}}_G^\varepsilon$ corresponds to the expression of $\boldsymbol{\beta}_G^\varepsilon$ in the $(\tilde{r}, \tilde{v}_r, t)$ coordinate system.

2.3 Noether's Theorem within this framework

As already said in the introduction, the dynamical system (1.6)-(1.7) is a perturbation of the dynamical system (1.11)-(1.12) and the (μ, θ) coordinate system is well adapted for the study of the dynamical system (1.11)-(1.12). The main argument discussed in the introduction was that in this coordinate system μ is an invariant of the trajectory. We will see in the next subsection that the Poincaré Cartan one-form associated with the dynamical system (1.6)-(1.7) is also a perturbation of the Poincaré Cartan one form associated with the dynamical system (1.11)-(1.12). Moreover, we will see that the non-exact part of the Poincaré Cartan one form associated with the dynamical system (1.11)-(1.12) does not depend on θ and consequently that it is invariant under the action of the flow of $\frac{\partial}{\partial \theta}$. Such flows are called symmetries of the Poincaré Cartan one form. The Noether's theorem connects such symmetries with invariants of the trajectory. Applying this Theorem in our case gives that $-\mu$ is the invariant corresponding to the flow of $\frac{\partial}{\partial \theta}$. Since the Poincaré Cartan one-form associated with the dynamical system (1.6)-(1.7) is a perturbation of the Poincaré Cartan one form associated with the dynamical system (1.11)-(1.12), the lowest order (in ε) of this one form, expressed in the (μ, θ) coordinate system, does not depend on θ. As a consequence, the flow of $\frac{\partial}{\partial \theta}$ is close to a symmetry. The goal of the Lie transform method, that we will introduce in the next section, is to find a coordinate system $(\tilde{\mu}, \tilde{\theta})$ close to the (μ, θ) coordinate system in which the flow of $\frac{\partial}{\partial \tilde{\theta}}$ is a symmetry and in which $-\tilde{\mu}$ is the corresponding invariant. The aim of this part is to introduce rigorously, within the framework of the Poincaré Cartan one form, these notions of symmetries, invariants and Noether's Theorem. The notions of symmetries and Noether's theorem can be written under a lot of forms. Indeed, there exists a lot of mathematical frameworks to study an Hamiltonian differential system and each of them provides an other formulation of the Noether's theorem. Nevertheless, in each of these mathematical frameworks a symmetry is a diffeomorphism, or a group of diffeomorphisms, leaving unchanged the principal object of the theory and the Noether's theorem connects these symmetries with the invariants of the trajectory. In this paper, according to Theorem 2.3, the principal object of the theory is the Poincaré-Cartan one form's equivalence class. Consequently, we will give the following definition of symmetries:

Definition 2.4. *Let $\mathcal{Y}$ be a vector field, $\mathcal{G}_\lambda$ its flow, and $\boldsymbol{\gamma}_G^\varepsilon$ the Poincaré Cartan one form associated with the dynamical system* (2.1)-(2.2)*. We will say that* $(\mathcal{G}_\lambda)$ *is a symmetry of* $[\boldsymbol{\gamma}_G^\varepsilon]$ *if for any λ for which $\mathcal{G}_\lambda$ is defined, $\mathcal{G}_\lambda^\star \boldsymbol{\gamma}_G^\varepsilon \in [\boldsymbol{\gamma}_G^\varepsilon]$; i.e., if $\mathcal{G}_\lambda^\star \boldsymbol{\gamma}_G^\varepsilon - \boldsymbol{\gamma}_G^\varepsilon$ is exact.*

This definition is well-posed with respect to the equivalence relation. Indeed, if $\boldsymbol{\beta}_G^\varepsilon \in [\boldsymbol{\gamma}_G^\varepsilon]$, then there exists a smooth function S^ε such that $\boldsymbol{\beta}_G^\varepsilon = \boldsymbol{\gamma}_G^\varepsilon + dS^\varepsilon$ and consequently if $\mathcal{G}_\lambda$ is a symmetry

$$\begin{aligned} \mathcal{G}_\lambda^\star \boldsymbol{\beta}_G^\varepsilon &= \mathcal{G}_\lambda^\star \left(\boldsymbol{\beta}_G^\varepsilon + dS^\varepsilon\right) \\ &= \mathcal{G}_\lambda^\star \boldsymbol{\gamma}_G^\varepsilon + d\mathcal{G}_\lambda^\star S^\varepsilon \\ &\in [\boldsymbol{\gamma}_G^\varepsilon] \,. \end{aligned} \tag{2.32}$$

Remark 2.1. *Easy computations lead to the fact that this definition of symmetry is well posed with respect to differential structure.*

On an other hand, a symmetry can be characterized by using directly the vector field that generates it.

Proposition 2.2. *Let $\mathcal{Y}$ be a vector field and $\mathcal{G}_\lambda$ its flow. Then, $\mathcal{G}_\lambda$ is a symmetry of $[\boldsymbol{\gamma}_G^\varepsilon]$ if and only if $\mathcal{L}_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon$ is exact.*

Proof. Assume that $\mathcal{G}_\lambda$ is a symmetry of $[\boldsymbol{\gamma}_G^\varepsilon]$. Then, there exists a smooth function P_λ^ε such that $\mathcal{G}_\lambda^\star \boldsymbol{\gamma}_G^\varepsilon - \boldsymbol{\gamma}_G^\varepsilon = dP_\lambda^\varepsilon$. As $\mathcal{G}_0^\star \boldsymbol{\gamma}_G^\varepsilon = \boldsymbol{\gamma}_G^\varepsilon$, there exists a smooth function Q_λ^ε such that $\mathcal{G}_\lambda^\star \boldsymbol{\gamma}_G^\varepsilon - \boldsymbol{\gamma}_G^\varepsilon = \lambda dQ_\lambda^\varepsilon$. By definition of the Lie derivative, $\mathcal{L}_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon = \frac{\partial \mathcal{G}_\lambda^\star \boldsymbol{\gamma}_G^\varepsilon}{\partial \lambda} \mid_{\lambda=0} = dQ_0^\varepsilon$ and consequently $\mathcal{L}_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon$ is exact.

Reciprocally, if $\mathcal{L}_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon$ is exact; i.e. if there exists a smooth function R^ε such that $\mathcal{L}_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon = dR^\varepsilon$, then, the usual formula

$$\frac{\partial}{\partial \lambda} \mathcal{G}_\lambda^\star \boldsymbol{\gamma}_G^\varepsilon \mid_{\lambda=\lambda_0} = \mathcal{G}_{\lambda_0}^\star \left(di_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon + i_{\boldsymbol{\mathcal{Y}}} d\boldsymbol{\gamma}_G^\varepsilon\right) \tag{2.33}$$

and the Cartan formula

$$\mathcal{L}_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon = di_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon + i_{\boldsymbol{\mathcal{Y}}} d\boldsymbol{\gamma}_G^\varepsilon \tag{2.34}$$

yield:

$$\begin{aligned} \frac{\partial}{\partial \lambda} \mathcal{G}_\lambda^\star \boldsymbol{\gamma}_G^\varepsilon &= \mathcal{G}_\lambda^\star \left(\mathcal{L}_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon\right) \\ &= \mathcal{G}_\lambda^\star \left(dR^\varepsilon\right) \\ &= d\left(\mathcal{G}_\lambda^\star R^\varepsilon\right). \end{aligned} \tag{2.35}$$

Finally an integration yields the result.

□

Now, we turn back to the notion of invariant.

Definition 2.5. *Let $\mathcal{I}$ be a smooth function on $\mathcal{M}$. We say that $\mathcal{I}$ is an invariant of* (2.1)-(2.2) *if and only if $i_{\boldsymbol{\tau}_G^\varepsilon} d\mathcal{I} = 0$.*

Remark 2.2. *Easy computations lead to the fact that this definition of invariant is well posed with respect to the differential structure.*

Having this material in hands, we can easily derive the Noether theorem within this framework.

Theorem 2.4. *Let $\boldsymbol{\mathcal{Y}}$ be a smooth vector field whose flow is a symmetry of $[\boldsymbol{\gamma}_G^\varepsilon]$. Let S^ε be a smooth function such that $\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon = dS^\varepsilon$. Then, $i_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon - S^\varepsilon$ is an invariant.*

Proof. The Cartan formula yields that $\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon = dS^\varepsilon$ is equivalent to

$$i_{\boldsymbol{\mathcal{Y}}} d\boldsymbol{\gamma}_G^\varepsilon + d i_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon = dS^\varepsilon. \tag{2.36}$$

Moreover, as $i_{\boldsymbol{\tau}_G^\varepsilon} d\boldsymbol{\gamma}_G^\varepsilon = 0$ we obtain:

$$\begin{aligned} i_{\boldsymbol{\tau}_G^\varepsilon} i_{\boldsymbol{\mathcal{Y}}} d\boldsymbol{\gamma}_G^\varepsilon &= \langle d\boldsymbol{\gamma}_G^\varepsilon ; \boldsymbol{\mathcal{Y}}, \boldsymbol{\tau}_G^\varepsilon \rangle \\ &= - \langle d\boldsymbol{\gamma}_G^\varepsilon ; \boldsymbol{\tau}_G^\varepsilon, \boldsymbol{\mathcal{Y}} \rangle \\ &= - \left\langle i_{\boldsymbol{\tau}_G^\varepsilon} d\boldsymbol{\gamma}_G^\varepsilon ; \boldsymbol{\mathcal{Y}} \right\rangle \\ &= 0. \end{aligned} \tag{2.37}$$

Consequently, applying $i_{\boldsymbol{\tau}_G^\varepsilon}$ at the both sides of (2.36) yields $i_{\boldsymbol{\tau}_G^\varepsilon} d\left(i_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon - S^\varepsilon\right) = 0$; i.e., $i_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon - S^\varepsilon$ is an invariant. $\square$

Remark 2.3. *Notice that Theorem 2.4 is compatible with the relation of equivalence. Indeed, if $\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\gamma}_G^\varepsilon = dS^\varepsilon$, then for any smooth function σ^ε, $\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{Y}}} \left(\boldsymbol{\gamma}_G^\varepsilon + d\sigma^\varepsilon\right) = d\left(S^\varepsilon + \boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{Y}}} \sigma^\varepsilon\right)$. In other words $\boldsymbol{\mathcal{Y}}$ generates a symmetry of $\boldsymbol{\gamma}_G^\varepsilon + d\sigma^\varepsilon$. Moreover, the associated invariant is $\left(\boldsymbol{\gamma}_G^\varepsilon + d\sigma^\varepsilon\right) \cdot \boldsymbol{\mathcal{Y}} - \left(S + L_{\boldsymbol{\mathcal{Y}}} \sigma^\varepsilon\right) = \boldsymbol{\gamma}_G^\varepsilon \cdot \boldsymbol{\mathcal{Y}} - S^\varepsilon$; i.e. the same invariant as the invariant associated to $\boldsymbol{\gamma}^\varepsilon$.*

Remark 2.4. *Easy computations lead to the fact that this Theorem is well posed with respect to the differential structure.*

Remark 2.5. *Definition 2.4 is a non-standard formulation of symmetry. A more popular approach, in cases where G_ε does not depend on t, is via momentum map (see for instance [12] or [13]). Within such framework, taking place on the symplectic manifold $\left(\mathbb{R}^2, dr \wedge dv_r\right)$, a symmetry associated with dynamical system* (2.1)-(2.2) *is a flow ψ_t^F of an Hamiltonian vector field $\boldsymbol{\mathfrak{X}}_F$ satisfying $G_\varepsilon\left(\psi_t^F\left(r, v_r\right)\right) = G_\varepsilon\left(r, v_r\right)$ for any $\left(r, v_r\right) \in \mathbb{R}^2$. Constructing the vector field $\mathbf{X}_F$ on $\mathcal{M}$ by setting $\mathbf{X}_F = \boldsymbol{\mathfrak{X}}_F + 0 \cdot \partial_t$; i.e., $\mathbf{X}_F = \partial_{v_r} F \partial_r - \partial_r F \partial_{v_r}$, we observe that $\boldsymbol{\mathcal{L}}_{\mathbf{X}_F} \boldsymbol{\gamma}_G^\varepsilon = d\left(-F + i_{\mathbf{X}_F} \boldsymbol{\gamma}_G^\varepsilon\right)$. Hence, the flow of $\mathbf{X}_F$ is also a symmetry in the sense of definition 2.4. Notice that the corresponding invariant is well the momentum map F. Consequently definition 2.4 is well an extension of the classical definition of symmetry in cases where dynamical system* (2.1)-(2.2) *is non autonomous.*

2.4 Application at the differential system (1.6)-(1.7)

The non perturbed case (Dynamical system (1.11)-(1.12)**)**

The solution of (1.11)-(1.12) is given by

$$\begin{pmatrix} \mathbf{R}^{\varepsilon}_{\mathbf{Un}} \\ \mathbf{V}^{\varepsilon}_{\mathbf{r},\mathbf{Un}} \end{pmatrix} = e^{tN^{\varepsilon}} \begin{pmatrix} r \\ v_r \end{pmatrix}, \tag{2.38}$$

where

$$N^{\varepsilon} = \begin{bmatrix} 0 & -\frac{1}{\varepsilon} \\ \frac{1}{\varepsilon} & 0 \end{bmatrix} \text{ and } e^{tN^{\varepsilon}} = \begin{bmatrix} \cos\left(\frac{t}{\varepsilon}\right) & -\sin\left(\frac{t}{\varepsilon}\right) \\ \sin\left(\frac{t}{\varepsilon}\right) & \cos\left(\frac{t}{\varepsilon}\right) \end{bmatrix}.$$

According to formula (2.38), the trajectories are circle of radius $\sqrt{r^2+v_r^2}$. Under the change of coordinates (1.15)-(1.16) dynamical system (1.11)-(1.12) reads:

$$\frac{\partial \mathfrak{Mu}^{\varepsilon}_{\mathbf{Un}}}{\partial t} = 0, \qquad \mathfrak{Mu}^{\varepsilon}_{\mathbf{Un}}(0,\mu,\theta) = \mu, \tag{2.39}$$

$$\frac{\partial \mathbf{\Theta}^{\varepsilon}_{\mathbf{Un}}}{\partial t} = -\frac{1}{\varepsilon}, \qquad \mathbf{\Theta}^{\varepsilon}_{\mathbf{Un}}(0,\mu,\theta) = \theta. \tag{2.40}$$

Making the change of coordinates (1.15)-(1.16) in the Poincaré Cartan one form, defined by (2.19) and with $G_\varepsilon = H^{\mathbf{Un}}_{\varepsilon}$ given by (1.13), yields:

$$\begin{aligned} \bar{\mathbf{\Gamma}}^{\varepsilon}_{H^{\mathbf{Un}}} &= \sin(\theta)\cos(\theta)\,d\mu - 2\mu\sin^2(\theta)\,d\theta - \frac{\mu}{\varepsilon}dt \\ &= -\mu d\theta - \frac{\mu}{\varepsilon}dt + d\left(\mu\sin(\theta)\cos(\theta)\right) \\ &= \bar{\boldsymbol{\beta}}^{\varepsilon}_{H^{\mathbf{Un}}} + d\left(\mu\sin(\theta)\cos(\theta)\right). \end{aligned} \tag{2.41}$$

The flow of $\frac{\partial}{\partial\theta}$ reads:

$$\bar{\mathcal{G}}_\lambda(\mu,\theta,t) = (\mu,\lambda+\theta,t). \tag{2.42}$$

As $\mathcal{L}_{\frac{\partial}{\partial\theta}}\bar{\boldsymbol{\beta}}^{\varepsilon}_{H^{\mathbf{Un}}} = 0$, proposition 2.2 yields that $\mathcal{G}_\lambda$ is a symmetry and Noether Theorem (Theorem 2.4) yields that $-\mu$ is the corresponding invariant.

The perturbed case (Dynamical system (1.6)-(1.7)**)**

Making the change of coordinates (1.15)-(1.16) in the Poincaré-Cartan one form, defined by (2.19) and with $G_\varepsilon = H_\varepsilon$, where H_ε is defined by (1.8), yields:

$$\begin{aligned} \bar{\mathbf{\Gamma}}^{\varepsilon}_{H_\varepsilon} &= \sin(\theta)\cos(\theta)\,d\mu - 2\mu\sin^2(\theta)\,d\theta - \left(\frac{\mu}{\varepsilon} + \phi_\varepsilon\left(\sqrt{2\mu}\cos(\theta),t\right)\right)dt \\ &= -\mu d\theta - \left(\frac{\mu}{\varepsilon} + \phi_\varepsilon\left(\sqrt{2\mu}\cos(\theta),t\right)\right)dt + d\left(\mu\sin(\theta)\cos(\theta)\right) \\ &= \bar{\boldsymbol{\beta}}^{\varepsilon}_{H_\varepsilon} + d\left(\mu\sin(\theta)\cos(\theta)\right). \end{aligned} \tag{2.43}$$

We remark that $\bar{\boldsymbol{\beta}}^{\varepsilon}_{H}$ defined by (2.43) is a perturbation of $\bar{\boldsymbol{\beta}}^{\varepsilon}_{H^{\mathbf{Un}}}$ defined by (2.41). Moreover, in this case the symmetry is broken; i.e., $\bar{\mathcal{G}}_\lambda$ defined by (2.42) is no longer a symmetry.

2.5 Change of coordinates as the flow of a vector field

Change of coordinates in a one form

Let $\boldsymbol{\omega}$ be a one form defined on $\mathcal{M}$ and $\mathbf{\Omega}$ its expression in the (r, v_r, t) coordinate system. If $(r, v_r, t) \in \mathcal{M}$ and $\mathbf{u} \in \mathcal{TM}_{(r,v_r,t)}$, we will use the following notation for $\boldsymbol{\omega}$ evaluated at (r, v_r, t) and applied at $\mathbf{u}$:

$$\boldsymbol{\omega}_{(r,v_r,t)} \cdot \mathbf{u} =< \boldsymbol{\omega}(r, v_r, t); \mathbf{u} > . \tag{2.44}$$

Let $\psi \ : \ (r, v_r, t) \mapsto (\tilde{r}, \tilde{v}_r, t) = (\boldsymbol{\mathcal{P}}\psi(r, v_r, t), t)$ be a change of coordinates belonging in $\boldsymbol{\mathcal{A}}$ and $\tilde{\mathbf{\Omega}}$ the expression of $\boldsymbol{\omega}$ in the $(\tilde{r}, \tilde{v}_r, t)$ coordinate system. Then, $\tilde{\mathbf{\Omega}}$ is given by $(\psi^{-1})^{\star}\mathbf{\Omega}$, where $(\psi^{-1})^{\star}\mathbf{\Omega}$ is called the pullback of $\mathbf{\Omega}$ by ψ^{-1} and is computed as follow:

$$< \tilde{\mathbf{\Omega}}(\tilde{r}, \tilde{v}_r, t); \tilde{\mathbf{u}} >=< \mathbf{\Omega}(\psi^{-1}(\tilde{r}, \tilde{v}_r, t)); (d\psi^{-1})_{(\tilde{r},\tilde{v}_r,t)} \cdot \tilde{\mathbf{u}} > . \tag{2.45}$$

In term of coordinates, formula (2.45) means that $\tilde{\mathbf{\Omega}}(\tilde{r}, \tilde{v}_r, t)$ corresponds to the line vector

$$\begin{aligned} &\Big[\tilde{\mathbf{\Omega}}_1(\tilde{r}, \tilde{v}_r, t), \tilde{\mathbf{\Omega}}_2(\tilde{r}, \tilde{v}_r, t), \tilde{\mathbf{\Omega}}_3(\tilde{r}, \tilde{v}_r, t)\Big] \\ &= \big[\mathbf{\Omega}_1(\psi^{-1}(\tilde{r}, \tilde{v}_r, t)), \mathbf{\Omega}_2(\psi^{-1}(\tilde{r}, \tilde{v}_r, t)), \mathbf{\Omega}_3(\psi^{-1}(\tilde{r}, \tilde{v}_r, t))\big]\, \nabla_{(\tilde{r},\tilde{v}_r,t)}\psi^{-1}(\tilde{r}, \tilde{v}_r, t). \end{aligned} \tag{2.46}$$

Usually, we also use the notation:

$$\begin{aligned} \tilde{\mathbf{\Omega}}(\tilde{r}, \tilde{v}_r, t) &= (\psi^{-1})^{\star}\mathbf{\Omega}(\tilde{r}, \tilde{v}_r, t) \\ &= \tilde{\mathbf{\Omega}}_1(\tilde{r}, \tilde{v}_r, t)d\tilde{r} + \tilde{\mathbf{\Omega}}_2(\tilde{r}, \tilde{v}_r, t)d\tilde{v_r} + \tilde{\mathbf{\Omega}}_3(\tilde{r}, \tilde{v}_r, t)dt, \end{aligned} \tag{2.47}$$

where $\tilde{\mathbf{\Omega}}_1(\tilde{r}, \tilde{v}_r, t)$, $\tilde{\mathbf{\Omega}}_2(\tilde{r}, \tilde{v}_r, t)$ and $\tilde{\mathbf{\Omega}}_3(\tilde{r}, \tilde{v}_r, t)$ are given by formula (2.46).

Change of coordinates as the flow of a vector field.

Theorem 2.5. *Let $(\bar{r}, \bar{v}_r, t)$ be local coordinates on $\mathcal{M}$, $\boldsymbol{\mathcal{Z}}$ a vector field on $\mathcal{M}$ and $\boldsymbol{\omega}$ a one form on $\mathcal{M}$. Let $\bar{\mathbf{Z}}$ and $\bar{\mathbf{\Omega}}$ be their expressions in the $(\bar{r}, \bar{v}_r, t)$ coordinate system. Assume that the last coordinates of $\bar{\mathbf{Z}}$ is* 0*; i.e. that*

$$\bar{\mathbf{Z}}\,(\tilde{r}, \tilde{v}_r, t) = \bar{Z}^1\,(\tilde{r}, \tilde{v}_r, t)\,\partial_{\bar{r}} + \bar{Z}^2\,(\tilde{r}, \tilde{v}_r, t)\,\partial_{\bar{v}_r}. \tag{2.48}$$

Let $\bar{\varphi}_\varepsilon$ be its flow; i.e. the solution of

$$\frac{\partial \bar{\varphi}_\varepsilon^1}{\partial \varepsilon} = \bar{Z}^1(\bar{\varphi}_\varepsilon), \tag{2.49}$$

$$\frac{\partial \bar{\varphi}_\varepsilon^2}{\partial \varepsilon} = \bar{Z}^2(\bar{\varphi}_\varepsilon), \qquad \bar{\varphi}_0(\bar{r}, \bar{v}_r, t) = (\bar{r}, \bar{v}_r, t), \tag{2.50}$$

$$\frac{\partial \bar{\varphi}_\varepsilon^3}{\partial \varepsilon} = 0. \tag{2.51}$$

Then, under the change of coordinates $(\bar{r}, \bar{v}_r, t) \mapsto (\tilde{r}, \tilde{v}_r, t) = \bar{\varphi}_\varepsilon(\bar{r}, \bar{v}_r, t)$, the expression $\tilde{\mathbf{\Omega}}_\varepsilon$ of $\boldsymbol{\omega}$ in the $(\tilde{r}, \tilde{v}_r, t)$ coordinate system admits the following expansion:

$$\begin{aligned} \tilde{\mathbf{\Omega}}_\varepsilon(\tilde{r}, \tilde{v}_r, t) &= \bar{\mathbf{\Omega}}(\tilde{r}, \tilde{v}_r, t) - \varepsilon\mathcal{L}_{\bar{\mathbf{Z}}}\bar{\mathbf{\Omega}}(\tilde{r}, \tilde{v}_r, t) + \ldots + \frac{(-1)^n\,\varepsilon^n}{n!}\mathcal{L}_{\bar{\mathbf{Z}}}^n\bar{\mathbf{\Omega}}(\tilde{r}, \tilde{v}_r, t) \\ &\quad + \frac{\varepsilon^{n+1}}{n!}\int_0^1 (1-u)^n \frac{\partial^{n+1}\tilde{\mathbf{\Omega}}_\varepsilon}{\partial \varepsilon^{n+1}} \,|_{\varepsilon u}\, (\tilde{r}, \tilde{v}_r, t)du, \end{aligned} \tag{2.52}$$

where $\mathcal{L}^k_{\bar{\mathbf{Z}}}\bar{\mathbf{\Omega}}$ *is defined recursively for* $k \geq 1$ *by*

$$\mathcal{L}_{\bar{\mathbf{Z}}}\bar{\mathbf{\Omega}} = \left(\frac{\partial}{\partial \varepsilon}\left((\bar{\varphi}_\varepsilon)^\star \bar{\mathbf{\Omega}}\right)\right)|_{\varepsilon=0} \tag{2.53}$$

and

$$\mathcal{L}^{k+1}_{\bar{\mathbf{Z}}}\bar{\mathbf{\Omega}} = \mathcal{L}_{\bar{\mathbf{Z}}}\left(\mathcal{L}^{k}_{\bar{\mathbf{Z}}}\bar{\mathbf{\Omega}}\right). \tag{2.54}$$

Moreover, the change of coordinates admits the following expansion in power of ε *:*

$$\begin{aligned}\tilde{r} =& \bar{r} + \varepsilon \bar{Z}^1(\bar{r}, \bar{v}_r, t) + \ldots + \frac{\varepsilon^n}{n!}\left(\mathcal{L}^{n-1}_{\bar{\mathbf{Z}}}\bar{Z}^1\right)(\bar{r}, \bar{v}_r, t) \\ &+ \frac{\varepsilon^{n+1}}{n!}\int_0^1 (1-u)^n \frac{\partial^{n+1}\bar{\varphi}^1_\varepsilon}{\partial \varepsilon^{n+1}} |_{\varepsilon u}(\bar{r}, \bar{v}_r, t)\, du, \\ \tilde{v}_r =& \bar{v}_r + \varepsilon \bar{Z}^2(\bar{r}, \bar{v}_r, t) + \ldots + \frac{\varepsilon^n}{n!}\left(\mathcal{L}^{n-1}_{\bar{\mathbf{Z}}}\bar{Z}^2\right)(\bar{r}, \bar{v}_r, t) \\ &+ \frac{\varepsilon^{n+1}}{n!}\int_0^1 (1-u)^n \frac{\partial^{n+1}\bar{\varphi}^2_\varepsilon}{\partial \varepsilon^{n+1}} |_{\varepsilon u}(\bar{r}, \bar{v}_r, t)\, du,\end{aligned} \tag{2.55}$$

and the reciprocal change of coordinates admits the following expansion:

$$\begin{aligned}\bar{r} =& \tilde{r} - \varepsilon \bar{Z}^1(\tilde{r}, \tilde{v}_r, t) + \ldots + \frac{(-1)^n \varepsilon^n}{n!}\left(\mathcal{L}^{n-1}_{\bar{\mathbf{Z}}}\bar{Z}^1\right)(\tilde{r}, \tilde{v}_r, t) \\ &+ \frac{\varepsilon^{n+1}}{n!}\int_0^1 (1-u)^n \left(\frac{\partial^{n+1}}{\partial \varepsilon^{n+1}}\bar{\varphi}^1_{-\varepsilon}\right) |_{\varepsilon u}(\tilde{r}, \tilde{v}_r, t)\, du, \\ \bar{v}_r =& \tilde{v}_r - \varepsilon \bar{Z}^2(\tilde{r}, \tilde{v}_r, t) + \ldots + \frac{(-1)^n \varepsilon^n}{n!}\left(\mathcal{L}^{n-1}_{\bar{\mathbf{Z}}}\bar{Z}^2\right)(\tilde{r}, \tilde{v}_r, t) \\ &+ \frac{\varepsilon^{n+1}}{n!}\int_0^1 (1-u)^n \left(\frac{\partial^{n+1}}{\partial \varepsilon^{n+1}}\bar{\varphi}^2_{-\varepsilon}\right) |_{\varepsilon u}(\tilde{r}, \tilde{v}_r, t)\, du.\end{aligned} \tag{2.56}$$

Proof. Let $(\bar{r}, \bar{v}_r, t)$ be local coordinates on $\mathcal{M}$, $\boldsymbol{\mathcal{Z}}$ a vector field on $\mathcal{M}$, and $\boldsymbol{\omega}$ a one form on $\mathcal{M}$. Let $\bar{\mathbf{Z}}$ and $\bar{\mathbf{\Omega}}$ be their expressions in the $(\bar{r}, \bar{v}_r, t)$ coordinates. We assume that the last coordinates of $\bar{\mathbf{Z}}$ is 0; i.e. that

$$\bar{\mathbf{Z}} = \bar{Z}^1 \partial_{\bar{r}} + \bar{Z}^2 \partial_{\bar{v}_r}. \tag{2.57}$$

Let $\bar{\varphi}_\varepsilon$ be its flow; i.e. the solution of (2.49)-(2.51). According to formula (2.47), under the change of coordinates $(\bar{r}, \bar{v}_r, t) \mapsto (\tilde{r}, \tilde{v}_r, t) = \bar{\varphi}_\varepsilon(\bar{r}, \bar{v}_r, t)$, the expression of $\boldsymbol{\omega}$ in the $(\tilde{r}, \tilde{v}_r, t)$ coordinates is given by $\tilde{\mathbf{\Omega}}_\varepsilon = (\bar{\varphi}_\varepsilon^{-1})^\star \bar{\mathbf{\Omega}}$. A Taylor expansion in power of ε yields:

$$\begin{aligned}\tilde{\mathbf{\Omega}}_\varepsilon(\tilde{r}, \tilde{v}_r, t) = \tilde{\mathbf{\Omega}}_0(\tilde{r}, \tilde{v}_r, t) + \varepsilon \frac{\partial \tilde{\mathbf{\Omega}}_\varepsilon}{\partial \varepsilon} |_{\varepsilon=0}(\tilde{r}, \tilde{v}_r, t) + \ldots + \frac{\varepsilon^n}{n!}\frac{\partial^n \tilde{\mathbf{\Omega}}_\varepsilon}{\partial \varepsilon^n} |_{\varepsilon=0}(\tilde{r}, \tilde{v}_r, t) \\ + \frac{\varepsilon^{n+1}}{n!}\int_0^1 (1-u)^n \frac{\partial^{n+1}\tilde{\mathbf{\Omega}}_\varepsilon}{\partial \varepsilon^{n+1}} |_{\varepsilon u}(\tilde{r}, \tilde{v}_r, t) du.\end{aligned} \tag{2.58}$$

Notice that for each $k \in \{1, \ldots, n+1\}$ we have use the following notation:

$$\frac{\partial^k \tilde{\mathbf{\Omega}}_\varepsilon}{\partial \varepsilon^k} = \left[\frac{\partial^k \tilde{\mathbf{\Omega}}_\varepsilon^1}{\partial \varepsilon^k}, \frac{\partial^k \tilde{\mathbf{\Omega}}_\varepsilon^2}{\partial \varepsilon^k} \right]. \tag{2.59}$$

As $\tilde{\mathbf{\Omega}}_\varepsilon = (\bar{\varphi}_\varepsilon^{-1})^\star \bar{\mathbf{\Omega}}$, we have for each $k \in \{1, \ldots, n+1\}$

$$\frac{\partial^k \tilde{\mathbf{\Omega}}_\varepsilon}{\partial \varepsilon^k} \mid_{\varepsilon=0} = \left(\frac{\partial^k}{\partial \varepsilon^k} \left((\bar{\varphi}_\varepsilon^{-1})^\star \bar{\mathbf{\Omega}} \right) \right) \mid_{\varepsilon=0} . \tag{2.60}$$

By definition, the Lie derivative of $\bar{\mathbf{\Omega}}$ with respect to $-\bar{\mathbf{Z}}$ is given by

$$\mathcal{L}_{-\bar{\mathbf{Z}}} \bar{\mathbf{\Omega}} = \left(\frac{\partial}{\partial \varepsilon} \left((\bar{\varphi}_\varepsilon^{-1})^\star \bar{\mathbf{\Omega}} \right) \right) \mid_{\varepsilon=0} \tag{2.61}$$

and easy computations lead to

$$(-1)^k \mathcal{L}_{\bar{\mathbf{Z}}}^k \bar{\mathbf{\Omega}} = \mathcal{L}_{-\bar{\mathbf{Z}}}^k \bar{\mathbf{\Omega}} = \left(\frac{\partial^k}{\partial \varepsilon^k} \left((\bar{\varphi}_\varepsilon^{-1})^\star \bar{\mathbf{\Omega}} \right) \right) \mid_{\varepsilon=0}, \tag{2.62}$$

where $\mathcal{L}_{\bar{\mathbf{Z}}}^k \bar{\mathbf{\Omega}}$ is defined recursively by formulas (2.53)-(2.54). Injecting formulas (2.62) in (2.58) leads to formula (2.52).

In the same way, Taylor's expansions of the inverse of the flow; i.e. of $\bar{\varphi}_{-\varepsilon}$, and of the flow; i.e. $\bar{\varphi}_\varepsilon$, lead to formulas (2.55) and (2.56).

This ends the proof of Theorem 2.5.

□

3 Lie Transform Method

3.1 The Lie Change of Coordinates

Subsequently, we will denote by $\boldsymbol{\gamma}^\varepsilon$ the Poicarré-Cartan one form associated with the dynamical system (1.9)-(1.10). We will also denote by $\boldsymbol{\beta}^\varepsilon \in [\boldsymbol{\gamma}^\varepsilon]$ the one form whose expression in the (μ, θ, t) coordinate system, defined by (1.15)-(1.16), is given by (2.43);i.e., by

$$\bar{\boldsymbol{\beta}}^\varepsilon = -\mu d\theta - \left(\frac{\mu}{\varepsilon} + \phi_\varepsilon \left(\sqrt{2\mu} \cos(\theta), t \right) \right) dt. \tag{3.1}$$

Injecting the Hilbert expansions the electric potential, given by (1.29), in (3.1) leads to the following Hilbert expansion of $\bar{\boldsymbol{\beta}}^\varepsilon$:

$$\bar{\boldsymbol{\beta}}^\varepsilon = \frac{1}{\varepsilon} \left(\bar{\boldsymbol{\beta}}_0 + \varepsilon \bar{\boldsymbol{\beta}}_1 + \varepsilon^2 \bar{\boldsymbol{\beta}}_2 + \ldots \right), \tag{3.2}$$

where

$$\begin{aligned}
\bar{\boldsymbol{\beta}}_0 (\mu, \theta, t) &= -\mu dt, \\
\bar{\boldsymbol{\beta}}_1 (\mu, \theta, t) &= -\mu d\theta - \phi_0 \left(\sqrt{2\mu} \cos(\theta), t \right) dt, \\
\bar{\boldsymbol{\beta}}_2 (\mu, \theta, t) &= -\phi_1 \left(\sqrt{2\mu} \cos(\theta), t \right) dt, \\
&\vdots
\end{aligned} \tag{3.3}$$

According to definition 1.1, a Lie change of coordinate is a composite of flows of vector fields $\dots, \bar{\mathbf{Z}}^3, \bar{\mathbf{Z}}^2, \bar{\mathbf{Z}}^1$ parametrized by $\dots \varepsilon^3, \varepsilon^2, \varepsilon$. In the same way as in Theorem 2.5 we will give in the following Theorem an Hilbert expansion of the expression of $\boldsymbol{\beta}^\varepsilon$ in the Lie coordinate system. Notice that the expression of the Hilbert expansion of $\tilde{\boldsymbol{\beta}}^\varepsilon$ involves only the expressions of the vector fields $\bar{\mathbf{Z}}^1, \bar{\mathbf{Z}}^2, \bar{\mathbf{Z}}^3, \dots$ and the expressions of the terms of the Hilbert expansion of $\bar{\boldsymbol{\beta}}^\varepsilon$.

Theorem 3.1. *Let $\boldsymbol{\gamma}^\varepsilon$ be the one form whose expression in the (r, v_r, t) coordinate system is defined by* (2.19)*. Let $\boldsymbol{\beta}^\varepsilon \in [\boldsymbol{\gamma}^\varepsilon]$ be the one form whose expression in the (μ, θ, t) coordinate system, defined by* (1.15)*-*(1.16)*, is given by* (3.1)*. Let $\boldsymbol{\mathcal{L}}_\varepsilon : \ (\mu, \theta, t) \mapsto \left(\tilde{\mu}, \tilde{\theta}, t\right)$ be a Lie change of coordinates. Then the expression $\tilde{\boldsymbol{\beta}}^\varepsilon$ of $\boldsymbol{\beta}^\varepsilon$ in the Lie coordinates $\left(\tilde{\mu}, \tilde{\theta}, t\right)$ is given by:*

$$\tilde{\boldsymbol{\beta}}^\varepsilon \left(\tilde{\mu}, \tilde{\theta}, t\right) = \frac{1}{\varepsilon} \sum_{m \geq 0} \left(\sum_{k=0}^{m} \left(\bar{\boldsymbol{W}}_k \bar{\boldsymbol{\beta}}_{m-k} \right) \left(\tilde{\mu}, \tilde{\theta}, t\right) \right) \varepsilon^m, \tag{3.4}$$

where for each $k \in \mathbb{N}^\star$, $\bar{\boldsymbol{W}}_k$ is defined by

$$\bar{\boldsymbol{W}}_k = \sum_{n_1 + 2n_2 + \dots + kn_k = k} \frac{(-1)^{n_1} \dots (-1)^{n_k}}{n_1! \dots n_k!} \mathcal{L}_{\bar{\mathbf{Z}}^k}^{n_k} \dots \mathcal{L}_{\bar{\mathbf{Z}}^1}^{n_1} \tag{3.5}$$

and $\bar{\boldsymbol{W}}_0 = id$. Moreover, the change of coordinates admits the following expansion in power of ε :

$$\begin{aligned} \left(\tilde{\mu}, \tilde{\theta}, t\right) &= \boldsymbol{\mathcal{L}}_\varepsilon (\mu, \theta, t) \\ &= \left(\sum_{k \geq 0} \varepsilon^k \left(\sum_{n_1 + 2n_2 + \dots + kn_k = k} \frac{\left(\bar{\mathbf{Z}}^1\right)^{n_1} \dots \left(\bar{\mathbf{Z}}^k\right)^{n_k}}{n_1! \dots n_k!} \right) (id) \right) (\mu, \theta, t), \end{aligned} \tag{3.6}$$

and the reciprocal change of coordinates admits the following expansion:

$$\begin{aligned} (\mu, \theta, t) &= \boldsymbol{\mathcal{L}}_\varepsilon^{-1} \left(\tilde{\mu}, \tilde{\theta}, t\right) \\ &= \left(\sum_{k \geq 0} \varepsilon^k \left(\sum_{n_1 + 2n_2 + \dots + kn_k = k} \frac{\left(-\bar{\mathbf{Z}}^1\right)^{n_1} \dots \left(-\bar{\mathbf{Z}}^k\right)^{n_k}}{n_1! \dots n_k!} \right) (id) \right) \left(\tilde{\mu}, \tilde{\theta}, t\right). \end{aligned} \tag{3.7}$$

Proof. We will start the proof by proving formulas (3.6) and (3.7). Let $g = g(\mu, \theta, t)$ be a smooth function,

$$\mathbf{v} = \xi^1 \partial_\mu + \xi^2 \partial_\theta + \xi^3 \partial_t \tag{3.8}$$

a smooth vector field and $\varphi_\varepsilon^{\mathbf{v}}$ its flow. Then, $(\varphi_\varepsilon^{\mathbf{v}})^\star g = g \circ \varphi_\varepsilon^{\mathbf{v}}$ admits the following Taylor expansion:

$$((\varphi_\varepsilon^{\mathbf{v}})^\star g)(\mu,\theta,t) = g(\mu,\theta,t) + \varepsilon(\mathbf{v}\cdot g)(\mu,\theta,t) + \ldots + \frac{\varepsilon^n}{n!}(\mathbf{v}^n \cdot g)(\mu,\theta,t) \tag{3.9}$$

$$+ \frac{\varepsilon^{n+1}}{n!}\int_0^1 (1-u)^n \frac{\partial^{n+1}}{\partial \varepsilon^{n+1}}\left(g\left(\varphi_\varepsilon^{\mathbf{v}}(\mu,\theta,t)\right)\right)|_{\varepsilon u}\, du, \tag{3.10}$$

where $\mathbf{v}\cdot g = \xi^1\partial_\mu g + \xi^2\partial_\theta g + \xi^3\partial_t g$ and $\mathbf{v}^{k+1}\cdot g = \mathbf{v}\cdot(\mathbf{v}^k \cdot g)$. Writing formally the entire Taylor series in ε, we obtain:

$$((\varphi_\varepsilon^{\mathbf{v}})^\star g)(\mu,\theta,t) = \left(\left(\sum_{n\geq 0}\frac{\varepsilon^n}{n!}\mathbf{v}^k\cdot\right) g\right)(\mu,\theta,t). \tag{3.11}$$

The right hand side of (3.11) is usually called the Lie series for the action of the flow on g. The same result hold for vector valued function $G: \ \mathcal{M} \to \mathbb{R}^m, G = (G^1,\ldots,G^m)$, where we let $\mathbf{v}$ act component-wise on $G: \mathbf{v}\cdot G = (\mathbf{v}\cdot G^1,\ldots,\mathbf{v}\cdot G^m)$. In our case, the change of coordinates reads:

$$\begin{aligned}
\left(\tilde{\mu},\tilde{\theta},t\right) &= \boldsymbol{\mathcal{L}}_\varepsilon(\mu,\theta,t) \\
&= \ldots \circ \bar{\varphi}^n_{\varepsilon^n} \circ \ldots \circ \bar{\varphi}^1_{\varepsilon^1}(\mu,\theta,t) \\
&= \left(\left(\ldots \circ \bar{\varphi}^n_{\varepsilon^n} \circ \ldots \circ \bar{\varphi}^1_{\varepsilon^1}\right)^\star (id)\right)(\mu,\theta,t) \\
&= \left(\left(\left(\bar{\varphi}^1_{\varepsilon^1}\right)^\star \circ \ldots \circ \left(\bar{\varphi}^n_{\varepsilon^n}\right)^\star \circ \ldots\right)(id)\right)(\mu,\theta,t).
\end{aligned} \tag{3.12}$$

According to formula (3.11), we have for each $n \in \mathbb{N}$ and for each vector valued function G :

$$\left(\left(\bar{\varphi}^k_{\varepsilon^k}\right)^\star G\right)(\mu,\theta,t) = \left(\left(\sum_{n_k\geq 0}\frac{\varepsilon^{kn_k}}{n_k!}(\bar{\mathbf{Z}}^k)^{n_k}\cdot\right) G\right)(\mu,\theta,t). \tag{3.13}$$

As a consequence, formula (3.12) can be rewritten:

$$\begin{aligned}
\boldsymbol{\mathcal{L}}_\varepsilon(\mu,\theta,t) &= \left(\left(\left(\sum_{n_1\geq 0}\frac{\varepsilon^{n_1}}{n_1!}\left(\bar{\mathbf{Z}}^1\right)^{n_1}\right)\cdot\left(\sum_{n_1\geq 0}\frac{\varepsilon^{2n_2}}{n_2!}\left(\bar{\mathbf{Z}}^2\right)^{n_2}\right)\cdot\ldots\right)(id)\right)(\mu,\theta,t) \\
&= \left(\left(\sum_{n_1,n_2,n_3,\ldots\geq 0}\frac{\varepsilon^{n_1+2n_2+\ldots}\left(\bar{\mathbf{Z}}^1\right)^{n_1}\left(\bar{\mathbf{Z}}^2\right)^{n_2}\ldots}{n_1!n_2!\ldots}\cdot\right)(id)\right)(\mu,\theta,t).
\end{aligned} \tag{3.14}$$

Grouping together the terms with the same power of ε leads to formula (3.6). In the same way we obtain formula (3.7).

Now, we will prove formula (3.4). Let $\bar{\mathbf{\Omega}} = \bar{\mathbf{\Omega}}(\mu,\theta,t)$ be a differential one-form,

$$\mathbf{w} = w^1\partial_\mu + w^2\partial_\theta \tag{3.15}$$

a smooth vector field, and $\varphi_\varepsilon^{\mathbf{w}}$ its flow. Then, according to Theorem 2.5, $\left((\varphi_\varepsilon^{\mathbf{v}})^{-1}\right)^\star \bar{\boldsymbol{\Omega}}$ admits the following Taylor expansion:

$$\begin{aligned}\left(\left((\varphi_\varepsilon^{\mathbf{w}})^{-1}\right)^\star \bar{\boldsymbol{\Omega}}\right)\left(\tilde{\mu},\tilde{\theta},t\right) &= \bar{\boldsymbol{\Omega}}\left(\tilde{\mu},\tilde{\theta},t\right) - \varepsilon \mathcal{L}_{\mathbf{w}} \bar{\boldsymbol{\Omega}}\left(\tilde{\mu},\tilde{\theta},t\right) + \ldots \\ &+ \frac{(-1)^n \varepsilon^n}{n!} \mathcal{L}_{\mathbf{w}}^n \bar{\boldsymbol{\Omega}}\left(\tilde{\mu},\tilde{\theta},t\right) + \frac{\varepsilon^{n+1}}{n!}\int_0^1 (1-u)^n \frac{\partial^{n+1}\tilde{\boldsymbol{\Omega}}_\varepsilon}{\partial \varepsilon^{n+1}}\,|_{\varepsilon u}\left(\tilde{\mu},\tilde{\theta},t\right) du\end{aligned} \tag{3.16}$$

Writing formally the entire Taylor series in ε, we obtain:

$$\left(\left((\varphi_\varepsilon^{\mathbf{w}})^{-1}\right)^\star \bar{\boldsymbol{\Omega}}\right)\left(\tilde{\mu},\tilde{\theta},t\right) = \left(\left(\sum_{n\geq 0} \frac{(-1)^n \varepsilon^n}{n!}\mathcal{L}_{\mathbf{w}}^n\right)\bar{\boldsymbol{\Omega}}\right)\left(\tilde{\mu},\tilde{\theta},t\right). \tag{3.17}$$

The right hand side of (3.17) is usually called the Lie series for the action of the flow on $\bar{\boldsymbol{\Omega}}$. Now, according to formula (2.47), the expression of $\boldsymbol{\beta}^\varepsilon$ in the Lie coordinate system is given by

$$\tilde{\boldsymbol{\beta}}^\varepsilon = \left(\boldsymbol{\mathcal{L}}_\varepsilon^{-1}\right)^\star \bar{\boldsymbol{\beta}}^\varepsilon. \tag{3.18}$$

Injecting (3.2) in (3.18) leads to:

$$\tilde{\boldsymbol{\beta}}^\varepsilon = \sum_{p\geq 0} \varepsilon^{p-1}\left(\boldsymbol{\mathcal{L}}_\varepsilon^{-1}\right)^\star \bar{\boldsymbol{\beta}}_p. \tag{3.19}$$

Consequently we obtain for each $p \in \mathbb{N}$:

$$\begin{aligned}&\left(\left(\boldsymbol{\mathcal{L}}_\varepsilon^{-1}\right)^\star \bar{\boldsymbol{\beta}}_p\right)\left(\tilde{\mu},\tilde{\theta},t\right) \\ &= \left(\left(\left(\ldots \circ \bar{\varphi}^n_{\varepsilon^n} \circ \ldots \circ \bar{\varphi}^1_{\varepsilon^1}\right)^{-1}\right)^\star \bar{\boldsymbol{\beta}}_p\right)\left(\tilde{\mu},\tilde{\theta},t\right) \\ &= \left(\left(\ldots \circ \left((\bar{\varphi}^n_{\varepsilon^n})^{-1}\right)^\star \circ \ldots \circ \left((\bar{\varphi}^1_{\varepsilon^1})^{-1}\right)^\star\right)\bar{\boldsymbol{\beta}}_p\right)\left(\tilde{\mu},\tilde{\theta},t\right) \\ &= \left(\left(\ldots\left(\sum_{n_2\geq 0}\frac{(-1)^{n_2}\varepsilon^{n_2}}{n_2!}\mathcal{L}^{n_2}_{\bar{\mathbf{Z}}^2}\right)\left(\sum_{n_1\geq 0}\frac{(-1)^{n_1}\varepsilon^{n_1}}{n_1!}\mathcal{L}^{n_1}_{\bar{\mathbf{Z}}^1}\right)\right)\bar{\boldsymbol{\beta}}_p\right)\left(\tilde{\mu},\tilde{\theta},t\right) \\ &= \left(\sum_{k\geq 0}\varepsilon^k\left(\sum_{n_1+2n_2+\ldots+kn_k=k}(-1)^{n_1+\ldots+n_k}\frac{\mathcal{L}^{n_k}_{\bar{\mathbf{Z}}^k}\ldots\mathcal{L}^{n_1}_{\bar{\mathbf{Z}}^1}}{n_1!\ldots n_k!}\right)\bar{\boldsymbol{\beta}}_p\right)\left(\tilde{\mu},\tilde{\theta},t\right).\end{aligned} \tag{3.20}$$

Injecting (3.20) in (3.19) and grouping together the terms with the same power of ε leads to formula (3.4).

This ends the proof of Theorem 3.1. □

We will denote by $\tilde{\boldsymbol{\beta}}_n$ the $(n-1)$th order of the Hilbert expansion (3.4); i.e.,

$$\tilde{\boldsymbol{\beta}}_n\left(\tilde{\mu},\tilde{\theta},t\right) = \sum_{k=0}^{n}\left(\bar{\boldsymbol{W}}_k \bar{\boldsymbol{\beta}}_{n-k}\right)\left(\tilde{\mu},\tilde{\theta},t\right). \tag{3.21}$$

3.2 The Lie Transform Method

The Lie Transform method consists to find a differential one form $\tilde{\boldsymbol{\alpha}}_\varepsilon \in [\tilde{\boldsymbol{\gamma}}^\varepsilon]$ and a Lie change of coordinates $\boldsymbol{\mathcal{L}}_\varepsilon$ such that $\tilde{\boldsymbol{\alpha}}_\varepsilon$ is under a normal form. We will precise immediately our definition of normal forms. For this purpose, we will introduce the following linear spaces of smooth functions:

$$\mathcal{C}_{2\pi}^\infty = \{f \in \mathcal{C}^\infty (\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}) ;\ f \text{ is } 2\pi \text{ periodic with respect to } \theta\}, \tag{3.22}$$

$$\mathcal{D} = \left\{ f \in \mathcal{C}_{2\pi};\ \frac{\partial f}{\partial \theta} = 0 \right\}, \tag{3.23}$$

$$\mathcal{R} = \left\{ f \in \mathcal{C}_{2\pi}^\infty;\ \langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\mu, \theta)\, d\theta = 0 \right\}. \tag{3.24}$$

Notice also that $\mathcal{C}_{2\pi}^\infty = \mathcal{D} \oplus \mathcal{R}$.

Definition 3.1. *Let $\boldsymbol{\mathcal{L}}_\varepsilon : (\mu, \theta, t) \mapsto \left(\tilde{\mu}, \tilde{\theta}, t\right)$ be a Lie change of coordinates, $\tilde{\boldsymbol{\alpha}}_\varepsilon = \tilde{\boldsymbol{\alpha}}_\varepsilon \left(\tilde{\mu}, \tilde{\theta}, t\right)$ be a differential one form admitting a Hilbert expansion of the form:*

$$\tilde{\boldsymbol{\alpha}}_\varepsilon = \frac{1}{\varepsilon} \sum_{n \geq 0} \left(\tilde{\alpha}_n^1 d\tilde{\mu} + \tilde{\alpha}_n^2 d\tilde{\theta} + \tilde{\alpha}_n^3 dt \right) \varepsilon^n, \tag{3.25}$$

and $\boldsymbol{\alpha}_\varepsilon = \boldsymbol{\alpha}_\varepsilon (\mu, \theta, t)$ the differential one form defined by $\boldsymbol{\alpha}_\varepsilon (\mu, \theta, t) = \tilde{\boldsymbol{\alpha}}_\varepsilon (\mu, \theta, t)$. We say that $\tilde{\boldsymbol{\alpha}}_\varepsilon$ is under a normal form if

$$\forall n \in \mathbb{N},\ \alpha_n^1 \in \mathcal{D}, \tag{3.26}$$

$$\alpha_1^2 = -\mu,\ \textit{and}\ \forall n \in \mathbb{N} \setminus \{1\},\ \alpha_n^2 = 0, \tag{3.27}$$

$$\forall n \in \mathbb{N},\ \alpha_n^3 \in \mathcal{D}. \tag{3.28}$$

This definition is made in order to have the following theorem:

Theorem 3.2. *Let $\boldsymbol{\mathcal{L}}_\varepsilon : (\mu, \theta, t) \mapsto \left(\tilde{\mu}, \tilde{\theta}, t\right)$ be a Lie change of coordinates and $\tilde{\boldsymbol{X}}_H^\varepsilon$ the expression of $\boldsymbol{\tau}^\varepsilon$ in the Lie coordinate system. Assume that there exists $\tilde{\boldsymbol{\alpha}}_\varepsilon \in [\tilde{\boldsymbol{\gamma}}^\varepsilon]$ which is under a normal form. Then, the first component of $\tilde{\mathbf{X}}_H^\varepsilon$ vanish, the second component is $\tilde{\theta}$ independent and it is given by*

$$\left(\tilde{X}_H^\varepsilon\right)^2 = \frac{\partial \tilde{\alpha}_\varepsilon^3}{\partial \tilde{\mu}} - \frac{\partial \tilde{\alpha}_\varepsilon^1}{\partial t}, \tag{3.29}$$

and the expression of the particle distribution in the Lie coordinate system satisfies:

$$\frac{\partial \tilde{f}_\varepsilon}{\partial t} \left(\tilde{\mu}, \tilde{\theta}, t\right) + \left(\tilde{X}_H^\varepsilon\right)^2 (\tilde{\mu}, t) \frac{\partial \tilde{f}_\varepsilon}{\partial t} \left(\tilde{\mu}, \tilde{\theta}, t\right) = 0. \tag{3.30}$$

Proof. Let $\boldsymbol{\mathcal{L}}_\varepsilon:\ (\mu,\theta,t)\mapsto\left(\tilde\mu,\tilde\theta,t\right)$ be a Lie change of coordinates and $\tilde{\boldsymbol{\alpha}}_\varepsilon\in[\tilde{\boldsymbol{\gamma}}^\varepsilon]$ which is under a normal form. According to Theorem 2.3 the expression of $\boldsymbol{\tau}^\varepsilon$ in the Lie coordinate system corresponds to the solution of

$$i_{\tilde{\mathbf{X}}_H^\varepsilon}d\tilde{\boldsymbol{\alpha}}_\varepsilon=0 \tag{3.31}$$

satisfying $\left(\tilde X_H^\varepsilon\right)^3=1$. Since all the components of $\tilde{\boldsymbol{\alpha}}_\varepsilon$ belong to $\mathcal{D}$, its differential is given by:

$$d\tilde{\boldsymbol{\alpha}}_\varepsilon=\frac{\partial\tilde\alpha_\varepsilon^1}{\partial t}dt\wedge d\tilde\mu-d\tilde\mu\wedge d\tilde\theta+\frac{\partial\tilde\alpha_\varepsilon^3}{\partial\tilde\mu}d\tilde\mu\wedge dt, \tag{3.32}$$

and consequently

$$\begin{aligned}&i_{\tilde{\mathbf{X}}_H^\varepsilon}d\tilde{\boldsymbol{\alpha}}_\varepsilon\\ &=\left(\frac{\partial\tilde\alpha_\varepsilon^1}{\partial t}-\frac{\partial\tilde\alpha_\varepsilon^3}{\partial\tilde\mu}+\left(\tilde X_H^\varepsilon\right)^2\right)d\tilde\mu-\left(\tilde X_H^\varepsilon\right)^1d\tilde\theta+\left(\left(\tilde X_H^\varepsilon\right)^1\frac{\partial\tilde\alpha_\varepsilon^3}{\partial\tilde\mu}-\left(\tilde X_H^\varepsilon\right)^1\right)dt\end{aligned} \tag{3.33}$$

Since $\tilde{\mathbf{X}}_H^\varepsilon$ satisfies (3.31), we obtain:

$$\left(\tilde X_H^\varepsilon\right)^1=0, \tag{3.34}$$

$$\frac{\partial\tilde\alpha_\varepsilon^1}{\partial t}-\frac{\partial\tilde\alpha_\varepsilon^3}{\partial\tilde\mu}+\left(\tilde X_H^\varepsilon\right)^2=0. \tag{3.35}$$

Since $\tilde\alpha_\varepsilon^1$ and $\tilde\alpha_\varepsilon^3$ lie in $\in\mathcal{D}$, equation (3.35) implies that $\left(\tilde X_H^\varepsilon\right)^2$ belongs to $\mathcal{D}$.

According to Theorem 2.2 the Vlasov equation reads

$$\frac{\partial\tilde f_\varepsilon}{\partial t}\left(\tilde\mu,\tilde\theta,t\right)+\left(\tilde X_H^\varepsilon\right)^2(\tilde\mu,t)\,\frac{\partial\tilde f_\varepsilon}{\partial t}\left(\tilde\mu,\tilde\theta,t\right)=0. \tag{3.36}$$

This ends the proof of Theorem (3.2). □

Having this material in hand we can precise the objectives of the Lie Transform method. The Lie transform method consists to find a sequence $\left(\bar{\mathbf{Z}}^n\right)_{n\in\mathbb{N}^\star}$ of vector fields and a sequence $(\tilde{\boldsymbol{\alpha}}_n)_{n\in\mathbb{N}}$ of differential one forms such that under the Lie change of coordinates $\boldsymbol{\mathcal{L}}_\varepsilon$ associated with this sequence of vector field the differential one form $\tilde{\boldsymbol{\alpha}}_\varepsilon$ defined by

$$\tilde{\boldsymbol{\alpha}}_\varepsilon=\frac{1}{\varepsilon}\sum_{n\geq0}\tilde{\boldsymbol{\alpha}}_n\varepsilon^n \tag{3.37}$$

is under a normal form and belongs to $[\boldsymbol{\gamma}^\varepsilon]$.

More precisely, let $\boldsymbol{\beta}^\varepsilon\in[\boldsymbol{\gamma}^\varepsilon]$ be the one form whose expression in the (μ,θ,t) coordinate system, defined by (1.15)-(1.16), is given by formula (3.1) and whose formal expansion in power of ε is given by (3.3). Let $\boldsymbol{\mathcal{L}}_\varepsilon:\ (\mu,\theta,t)\mapsto\left(\tilde\mu,\tilde\theta,t\right)$ be the unknown Lie Change of

Coordinates and $\tilde{\boldsymbol{\beta}}^{\varepsilon}$ the expression of $\boldsymbol{\beta}^{\varepsilon}$ in this unknown Lie coordinate system. According to Proposition 3.1, $\tilde{\boldsymbol{\beta}}^{\varepsilon}$ admits the expansion in power of ε given by (3.4). The Lie Transform method consists to construct by induction the sequences of vector fields and differential one forms such that for each $n \in \mathbb{N}^{\star}$

$$\frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\alpha}}_0 + \varepsilon\tilde{\boldsymbol{\alpha}}_1 + \ldots + \varepsilon^n \tilde{\boldsymbol{\alpha}}_n\right) \in \left[\frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\beta}}_0 + \varepsilon\tilde{\boldsymbol{\beta}}_1 + \ldots + \varepsilon^n \tilde{\boldsymbol{\beta}}_n\right)\right] \tag{3.38}$$

and such that the differential one form

$$\tilde{\boldsymbol{\alpha}}_{\varepsilon}^n = \frac{1}{\varepsilon}\sum_{k=0}^{n} \tilde{\boldsymbol{\alpha}}_k \varepsilon^k \tag{3.39}$$

is under a normal form.

Notice that by construction a Lie change of coordinate is infinitesimal and consequently the first term of the sequence defining $\tilde{\boldsymbol{\alpha}}_{\varepsilon}$ is given by

$$\tilde{\boldsymbol{\alpha}}_0 = -\tilde{\mu} dt. \tag{3.40}$$

Now, the constructive proof of the following Theorem constitutes the Lie Transform algorithm.

Theorem 3.3. *There exists a Lie change of coordinates $\boldsymbol{\mathcal{L}}_{\varepsilon}$ and a differential one form $\tilde{\boldsymbol{\alpha}}_{\varepsilon}$ such that $\tilde{\boldsymbol{\alpha}}_{\varepsilon}$ belongs to $[\boldsymbol{\gamma}^{\varepsilon}]$ and is under a normal form. Moreover the proof of this Theorem constitutes a constructive algorithm to build $\boldsymbol{\mathcal{L}}_{\varepsilon}$ and $\tilde{\boldsymbol{\alpha}}_{\varepsilon}$.*

3.3 The Lie Transform Algorithm: proof of Theorem 3.3

Lemma 3.1. *For any $\left(\bar{\mathbf{Z}}^n\right)_{n\geq 2} \in \mathcal{C}_{2\pi}^{\infty}$ and $\bar{Z}_2^1 \in \mathcal{C}_{2\pi}^{\infty}$, setting*

$$\begin{aligned}\bar{\mathbf{Z}}^1(\mu,\theta,t) = &\left(\phi_0\left(\sqrt{2\mu}\cos(\theta), t\right) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_0\left(\sqrt{2\mu}\cos(\theta), t\right) d\theta\right)\partial_{\mu} \\ &+ \bar{Z}_2^1 \partial_{\theta}\end{aligned} \tag{3.41}$$

and

$$\tilde{\boldsymbol{\alpha}}_1 = -\tilde{\mu} d\tilde{\theta} - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_0\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right), t\right) d\tilde{\theta}\right) dt \tag{3.42}$$

yields that

$$\tilde{\boldsymbol{\alpha}}_{\varepsilon}^1 = \frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\alpha}}_0 + \varepsilon\tilde{\boldsymbol{\alpha}}_1\right) \in \left[\frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\beta}}_0 + \varepsilon\tilde{\boldsymbol{\beta}}_1\right)\right] \tag{3.43}$$

and that $\tilde{\boldsymbol{\alpha}}_{\varepsilon}^1$ is under a normal form.

Proof. Applying formula (3.21) with $n = 1$ yields:

$$\tilde{\boldsymbol{\beta}}_1\left(\tilde{\mu},\tilde{\theta},t\right) = \bar{\boldsymbol{W}}_0\bar{\boldsymbol{\beta}}_1\left(\tilde{\mu},\tilde{\theta},t\right) + \bar{\boldsymbol{W}}_1\bar{\boldsymbol{\beta}}_0\left(\tilde{\mu},\tilde{\theta},t\right). \tag{3.44}$$

Computing $\bar{\boldsymbol{W}}_1$ with formula (3.5) and using Cartan Formula yields:

$$\bar{\boldsymbol{W}}_1 = -i_{\bar{\mathbf{Z}}^1}d - di_{\bar{\mathbf{Z}}^1}. \tag{3.45}$$

According to (3.45), the only non-exact contribution of $\bar{\boldsymbol{W}}_1$ is given by $-i_{\bar{\mathbf{Z}}^1}d$. Consequently, we just have to find $\tilde{\boldsymbol{\alpha}}_1$, S_1 and $\bar{\mathbf{Z}}^1$ such that:

$$\tilde{\boldsymbol{\alpha}}_1\left(\tilde{\mu},\tilde{\theta},t\right) = \bar{\boldsymbol{\beta}}_1\left(\tilde{\mu},\tilde{\theta},t\right) - \left(i_{\bar{\mathbf{Z}}^1}d\bar{\boldsymbol{\beta}}_0\right)\left(\tilde{\mu},\tilde{\theta},t\right) + \left(dS_1\right)\left(\tilde{\mu},\tilde{\theta},t\right), \tag{3.46}$$

and such that $\tilde{\boldsymbol{\alpha}}^1_\varepsilon = \frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\alpha}}_0 + \varepsilon\tilde{\boldsymbol{\alpha}}_1\right)$ is under a normal form. Writing formula (3.46) in coordinates yields:

$$\begin{aligned}&\left(\frac{\partial S_1}{\partial\tilde{\mu}} - \tilde{\alpha}^1_1\right)d\tilde{\mu} + \left(\frac{\partial S_1}{\partial\tilde{\theta}} - \tilde{\alpha}^2_1 - \tilde{\mu}\right)d\tilde{\theta} + \\ &\left(\frac{\partial S_1}{\partial t} + \bar{Z}^1_1 - \phi_0\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right) - \tilde{\alpha}^3_1\right)dt = 0.\end{aligned} \tag{3.47}$$

Setting $\tilde{\alpha}^1_1 = 0,\ \tilde{\alpha}^2_1 = -\tilde{\mu},\ ,\tilde{\alpha}^3_1 = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_0\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right)d\tilde{\theta}$,

$$\bar{Z}^1_1\left(\tilde{\mu},\tilde{\theta},t\right) = \phi_0\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_0\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right)d\tilde{\theta} \tag{3.48}$$

and $S_1 = 0$ yields the result. This ends the proof of Lemma 3.1.

□

Theorem 3.4. *For any* $\left(\bar{\mathbf{Z}}^n\right)_{n\geq 3} \in \mathcal{C}^\infty_{2\pi}$, $\bar{Z}^{2,\mathcal{D}}_1 \in \mathcal{D}$ *and* $\bar{Z}^2_2 \in \mathcal{C}^\infty_{2\pi}$, *setting*

$$\begin{aligned}\bar{Z}^1_2 = &\frac{1}{\sqrt{2\mu}}\int_0^\theta \cos\left(\theta'\right)E_0\left(\sqrt{2\mu}\cos\left(\theta'\right),t\right)d\theta' \\ &- \frac{\theta}{2\pi\sqrt{2\mu}}\int_{-\pi}^{\pi}\cos\left(\theta'\right)E_0\left(\sqrt{2\mu}\cos\left(\theta'\right),t\right)d\theta',\end{aligned} \tag{3.49}$$

$$\bar{Z}^{2,\mathcal{R}}_1\left(\tilde{\mu},\tilde{\theta},t\right) = \varrho_2\left(\tilde{\mu},\tilde{\theta},t\right) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\varrho_2\left(\tilde{\mu},\tilde{\theta},t\right)d\tilde{\theta}, \tag{3.50}$$

where ϱ_2 *is defined by formula* (3.65), *and*

$$\tilde{\boldsymbol{\alpha}}_2 = \left(\bar{Z}^{2,\mathcal{D}}_1 - \frac{1}{2\pi}\int_{-\pi}^{\pi}\varrho_2\left(\tilde{\mu},\tilde{\theta},t\right)d\tilde{\theta}\right)dt \tag{3.51}$$

yields that

$$\tilde{\boldsymbol{\alpha}}^2_\varepsilon = \frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\alpha}}_0 + \varepsilon\tilde{\boldsymbol{\alpha}}_1 + \varepsilon^2\tilde{\boldsymbol{\alpha}}_2\right) \in \left[\frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\beta}}_0 + \varepsilon\tilde{\boldsymbol{\beta}}_1 + \varepsilon^2\tilde{\boldsymbol{\beta}}_2\right)\right] \tag{3.52}$$

and that $\tilde{\boldsymbol{\alpha}}^2_\varepsilon$ *is under a normal form.*

Proof. Applying formula (3.21) with $n = 2$ yields:

$$\tilde{\boldsymbol{\beta}}_2\left(\tilde{\mu},\tilde{\theta},t\right) = \bar{\boldsymbol{W}}_0\bar{\boldsymbol{\beta}}_2\left(\tilde{\mu},\tilde{\theta},t\right) + \bar{\boldsymbol{W}}_1\bar{\boldsymbol{\beta}}_1\left(\tilde{\mu},\tilde{\theta},t\right) + \bar{\boldsymbol{W}}_2\bar{\boldsymbol{\beta}}_0\left(\tilde{\mu},\tilde{\theta},t\right). \tag{3.53}$$

Computing $\bar{\boldsymbol{W}}_2$ with formula (3.5) and using Cartan Formula yields:

$$\bar{\boldsymbol{W}}_2 = -i_{\bar{\mathbf{Z}}^2}d - di_{\bar{\mathbf{Z}}^2} + \frac{1}{2}\left(i_{\bar{\mathbf{Z}}^1}di_{\bar{\mathbf{Z}}^1}d + di_{\bar{\mathbf{Z}}^1}\mathcal{L}_{\bar{\mathbf{Z}}^1}\right). \tag{3.54}$$

According to (3.54), the only non-exact contribution of $\bar{\boldsymbol{W}}_2$ is given by

$$-i_{\bar{\mathbf{Z}}^2}d + \frac{1}{2}i_{\bar{\mathbf{Z}}^1}di_{\bar{\mathbf{Z}}^1}d.$$

Consequently, we just have to find S_2, $\bar{Z}_2^1$ and $\bar{\mathbf{Z}}^2$ such that:

$$\begin{aligned}\tilde{\boldsymbol{\alpha}}_2\left(\tilde{\mu},\tilde{\theta},t\right) = \bar{\boldsymbol{\beta}}_2\left(\tilde{\mu},\tilde{\theta},t\right) &- \left(i_{\bar{\mathbf{Z}}^2}d\bar{\boldsymbol{\beta}}_0\right)\left(\tilde{\mu},\tilde{\theta},t\right) - \left(i_{\bar{\mathbf{Z}}^1}d\bar{\boldsymbol{\beta}}_1\right)\left(\tilde{\mu},\tilde{\theta},t\right)\\ &+ \frac{1}{2}\left(i_{\bar{\mathbf{Z}}^1}di_{\bar{\mathbf{Z}}^1}d\bar{\boldsymbol{\beta}}_0\right)\left(\tilde{\mu},\tilde{\theta},t\right) + \left(dS_2\right)\left(\tilde{\mu},\tilde{\theta},t\right).\end{aligned} \tag{3.55}$$

Writing the terms of formula (3.55) in coordinates yields:

$$\begin{aligned}&i_{\bar{\mathbf{Z}}^2}d\bar{\boldsymbol{\beta}}_0 = -\bar{Z}_1^2 dt,\\ &i_{\bar{\mathbf{Z}}^1}d\bar{\boldsymbol{\beta}}_1 = \bar{Z}_2^1 d\tilde{\mu} - \bar{Z}_1^1 d\tilde{\theta} + \frac{\partial\phi_0}{\partial r}\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right)\left(\bar{Z}_2^1\sqrt{2\tilde{\mu}}\sin\left(\tilde{\theta}\right) - \bar{Z}_1^1\frac{\cos\left(\tilde{\theta}\right)}{\sqrt{2\tilde{\mu}}}\right)dt,\\ &i_{\bar{\mathbf{Z}}^1}di_{\bar{\mathbf{Z}}^1}d\bar{\boldsymbol{\beta}}_0 = -\left(\bar{Z}_1^1\frac{\partial\bar{Z}_1^1}{\partial\tilde{\mu}} + \bar{Z}_2^1\frac{\partial\bar{Z}_1^1}{\partial\tilde{\theta}}\right)dt.\end{aligned} \tag{3.56}$$

Consequently, (3.55) reads:

$$\frac{\partial S_2}{\partial\tilde{\mu}} - \bar{Z}_2^1 - \tilde{\alpha}_2^1 = 0, \tag{3.57}$$

$$\frac{\partial S_2}{\partial\tilde{\theta}} + \bar{Z}_1^1 - \tilde{\alpha}_2^2 = 0, \tag{3.58}$$

and

$$\begin{aligned}&\bar{Z}_1^2 - \tilde{\alpha}_2^3 + \frac{\partial S_2}{\partial t} - \phi_1\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right)\\ &-\left(\bar{Z}_2^1\sqrt{2\tilde{\mu}}\sin\left(\tilde{\theta}\right) - \bar{Z}_1^1\frac{\cos\left(\tilde{\theta}\right)}{\sqrt{2\tilde{\mu}}}\right)\frac{\partial\phi_0}{\partial r}\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right)\\ &-\frac{1}{2}\left(\bar{Z}_1^1\frac{\partial\bar{Z}_1^1}{\partial\tilde{\mu}} + \bar{Z}_2^1\frac{\partial\bar{Z}_1^1}{\partial\tilde{\theta}}\right) = 0\end{aligned} \tag{3.59}$$

Since $\mathcal{C}^{\infty}_{2\pi} = \mathcal{D} \oplus \mathcal{R}$, we make the following decompositions:

$$S_2 = S_2^{\mathcal{D}} + S_2^{\mathcal{R}}, \tag{3.60}$$

$$\bar{Z}_2^1 = \bar{Z}_2^{1,\mathcal{D}} + \bar{Z}_2^{1,\mathcal{R}}, \tag{3.61}$$

$$\bar{Z}_1^2 = \bar{Z}_1^{2,\mathcal{D}} + \bar{Z}_1^{2,\mathcal{R}}. \tag{3.62}$$

Setting $\tilde{\alpha}_2^2 = 0$ in (3.58) implies

$$\frac{\partial S_2^{\mathcal{R}}}{\partial \tilde{\theta}} = -\bar{Z}_1^1. \tag{3.63}$$

Since $\bar{Z}_1^1 \in \mathcal{R}$, equation (3.63) has a solution in $\mathcal{R}$ and it is given by

$$S_2^{\mathcal{R}} = -\int_0^{\theta} \phi_0\left(\sqrt{2\mu}\cos\left(\theta'\right), t\right) d\theta' + \frac{\theta}{2\pi}\int_{-\pi}^{\pi} \phi_0\left(\sqrt{2\mu}\cos\left(\theta'\right), t\right) d\theta'. \tag{3.64}$$

Afterwards, setting $\bar{Z}_2^1 = \frac{\partial S_2^{\mathcal{R}}}{\partial \tilde{\mu}}$ (notice that this choice implies $\bar{Z}_2^1 = \bar{Z}_2^{1,\mathcal{R}}$) and $S_2^{\mathcal{D}} = 0$ in (3.57) implies $\tilde{\alpha}_2^1 = 0$.

Finally, let ϱ_2 be the function defined by

$$\begin{aligned}
\varrho_2\left(\tilde{\mu}, \tilde{\theta}, t\right) = &-\frac{\partial S_2^{\mathcal{R}}}{\partial t} + \phi_1\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right), t\right) \\
&+ \left(\bar{Z}_2^1\sqrt{2\tilde{\mu}}\sin\left(\tilde{\theta}\right) - \bar{Z}_1^1\frac{\cos\left(\tilde{\theta}\right)}{\sqrt{2\tilde{\mu}}}\right)\frac{\partial \phi_0}{\partial r}\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right), t\right) \\
&+ \frac{1}{2}\left(\bar{Z}_1^1\frac{\partial \bar{Z}_1^1}{\partial \tilde{\mu}} + \bar{Z}_2^1\frac{\partial \bar{Z}_1^1}{\partial \tilde{\theta}}\right).
\end{aligned} \tag{3.65}$$

Then, equation (3.59) reads:

$$\bar{Z}_1^2 - \tilde{\alpha}_2^3 - \varrho_2\left(\tilde{\mu}, \tilde{\theta}, t\right) = 0. \tag{3.66}$$

Setting

$$\bar{Z}_1^{2,\mathcal{R}}\left(\tilde{\mu}, \tilde{\theta}, t\right) = \varrho_2\left(\tilde{\mu}, \tilde{\theta}, t\right) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \varrho_2\left(\tilde{\mu}, \tilde{\theta}, t\right) d\tilde{\theta} \tag{3.67}$$

remove the $\tilde{\theta}$ dependency in $\tilde{\alpha}_2^3$.

$\square$

Remark 3.1. *Notice that at this level $\bar{Z}_1^{2,\mathcal{D}}$ is not fixed. But as soon as it will be fixed, $\tilde{\alpha}_2^3$ will also be fixed and will be equal to*

$$\tilde{\alpha}_2^3 = \bar{Z}_1^{2,\mathcal{D}} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \varrho_2\left(\tilde{\mu}, \tilde{\theta}, t\right) d\tilde{\theta} \in \mathcal{D}.$$

Theorem 3.5. *For any $n \geq 2$, for any sequence $\left(\bar{\mathbf{Z}}^k\right)_{k\geq n+1} \in \mathcal{C}^\infty_{2\pi}$, for any $\bar{Z}_1^{n,\mathcal{D}} \in \mathcal{D}$ and for any $\bar{Z}_2^n \in \mathcal{C}^\infty_{2\pi}$, there exists $\left(\bar{\mathbf{Z}}^k\right)_{k\leq n-1} \in \mathcal{C}^\infty_{2\pi}$, $\bar{Z}_1^{n,\mathcal{R}} \in \mathcal{R}$ and $(\boldsymbol{\alpha}_k)_{0\leq k\leq n}$ such that*

$$\tilde{\boldsymbol{\alpha}}_\varepsilon^n = \frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\alpha}}_0 + \varepsilon\tilde{\boldsymbol{\alpha}}_1 + \ldots + \varepsilon^n\tilde{\boldsymbol{\alpha}}_n\right) \in \left[\frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\beta}}_0 + \varepsilon\tilde{\boldsymbol{\beta}}_1 + \ldots + \varepsilon^n\tilde{\boldsymbol{\beta}}_n\right)\right] \tag{3.68}$$

and such that $\tilde{\boldsymbol{\alpha}}_\varepsilon^n$ is under a normal form.

Proof. We will prove Theorem 3.5 by induction. The case $n = 2$ was treated in Theorem 3.4. Consequently, we pass directly to the induction step.

Let $n \geq 3$. Assume that $\bar{\mathbf{Z}}^1, \bar{\mathbf{Z}}^2, \ldots, \bar{\mathbf{Z}}^{n-2} \in \mathcal{C}^\infty_{2\pi}$ and $\bar{Z}_1^{n-1,\mathcal{R}} \in \mathcal{R}$ are fixed in such a way that

$$\tilde{\boldsymbol{\alpha}}_\varepsilon^{n-1} = \frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\alpha}}_0 + \varepsilon\tilde{\boldsymbol{\alpha}}_1 + \ldots + \varepsilon^{n-1}\tilde{\boldsymbol{\alpha}}_{n-1}\right) \in \left[\frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\beta}}_0 + \varepsilon\tilde{\boldsymbol{\beta}}_1 + \ldots + \varepsilon^{n-1}\tilde{\boldsymbol{\beta}}_{n-1}\right)\right] \tag{3.69}$$

and $\tilde{\boldsymbol{\alpha}}_\varepsilon^{n-1}$ is under a normal form. We will find $\bar{Z}_2^{n-1} \in \mathcal{C}^\infty_{2\pi}$, $\bar{Z}_1^{n-1,\mathcal{D}} \in \mathcal{D}$ and $\bar{Z}_1^{n,\mathcal{R}} \in \mathcal{R}$ such that:

$$\tilde{\boldsymbol{\alpha}}_\varepsilon^n = \frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\alpha}}_0 + \varepsilon\tilde{\boldsymbol{\alpha}}_1 + \ldots + \varepsilon^n\tilde{\boldsymbol{\alpha}}_n\right) \in \left[\frac{1}{\varepsilon}\left(\tilde{\boldsymbol{\beta}}_0 + \varepsilon\tilde{\boldsymbol{\beta}}_1 + \ldots + \varepsilon^n\tilde{\boldsymbol{\beta}}_n\right)\right], \tag{3.70}$$

and such that $\tilde{\boldsymbol{\alpha}}_\varepsilon^n$ is under a normal form.

Formula (3.21) yields:

$$\tilde{\boldsymbol{\beta}}_n\left(\tilde{\mu}, \tilde{\theta}, t\right) = \sum_{k=0}^{n}\left(\bar{\boldsymbol{W}}_k\bar{\boldsymbol{\beta}}_{n-k}\right)\left(\tilde{\mu}, \tilde{\theta}, t\right),$$

where $\bar{\boldsymbol{W}}_n$ is given by (3.5). As in formula (3.5) (with $k = n$) $n_1 + 2n_2 + \ldots + nn_n = n$, the only term depending on $\bar{\mathbf{Z}}^n$ in $\bar{\boldsymbol{W}}_n\bar{\boldsymbol{\beta}}_0$ is $-\mathcal{L}_{\bar{\mathbf{Z}}^n}\bar{\boldsymbol{\beta}}_0$, and the only term depending on $\bar{\mathbf{Z}}^{n-1}$ is $\mathcal{L}_{\bar{\mathbf{Z}}^{n-1}}\mathcal{L}_{\bar{\mathbf{Z}}^1}\bar{\boldsymbol{\beta}}_0$, and as in formula (3.5) (with $k = n-1$) $n_1 + 2n_2 + \ldots + (n-1)n_{n-1} = n-1$, the only term depending on $\bar{\mathbf{Z}}^{n-1}$ in $\bar{\boldsymbol{W}}_{n-1}\bar{\boldsymbol{\beta}}_1$ is $-\mathcal{L}_{\bar{\mathbf{Z}}^{n-1}}\bar{\boldsymbol{\beta}}_1$. Consequently, the only terms in formula (3.21) depending on $\bar{\mathbf{Z}}^{n-1}$ and $\bar{\mathbf{Z}}^n$ are $-\mathcal{L}_{\bar{\mathbf{Z}}^n}\bar{\boldsymbol{\beta}}_0$, $\mathcal{L}_{\bar{\mathbf{Z}}^{n-1}}\mathcal{L}_{\bar{\mathbf{Z}}^1}\bar{\boldsymbol{\beta}}_0$ and $-\mathcal{L}_{\bar{\mathbf{Z}}^{n-1}}\bar{\boldsymbol{\beta}}_1$. Hence $\tilde{\boldsymbol{\beta}}_n$ reads:

$$\tilde{\boldsymbol{\beta}}_n = \bar{\boldsymbol{\beta}}_n - i_{\bar{\boldsymbol{Z}}^n}d\bar{\boldsymbol{\beta}}_0 - i_{\bar{\boldsymbol{Z}}^{n-1}}d\bar{\boldsymbol{\beta}}_1 + i_{\bar{\boldsymbol{Z}}^{n-1}}di_{\bar{\boldsymbol{Z}}^1}d\bar{\boldsymbol{\beta}}_0 + \psi^n\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right) + \text{ something exact.} \tag{3.71}$$

Consequently, we just have to find S_n, $\bar{Z}_2^{n-1} \in \mathcal{C}^\infty_{2\pi}$, $\bar{Z}_1^{n-1,\mathcal{D}} \in \mathcal{D}$ and $\bar{Z}_1^{n,\mathcal{R}} \in \mathcal{R}$ such that:

$$\tilde{\boldsymbol{\alpha}}_n = \bar{\boldsymbol{\beta}}_n - i_{\bar{\boldsymbol{Z}}^n}d\bar{\boldsymbol{\beta}}_0 - i_{\bar{\boldsymbol{Z}}^{n-1}}d\bar{\boldsymbol{\beta}}_1 + i_{\bar{\boldsymbol{Z}}^{n-1}}di_{\bar{\boldsymbol{Z}}^1}d\bar{\boldsymbol{\beta}}_0 + \psi^n\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right) + dS_n \tag{3.72}$$

Writing formula (3.72) in coordinates yields:

$$
\begin{aligned}
& i_{\bar{Z}^n} d\bar{\boldsymbol{\beta}}_0 = -\bar{Z}_1^n dt, \\
& i_{\bar{Z}^{n-1}} d\bar{\boldsymbol{\beta}}_1 = \bar{Z}_2^{n-1} d\tilde{\mu} - \bar{Z}_1^{n-1} d\tilde{\theta} + \frac{\partial \phi_0}{\partial r}\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right), t\right)\left(\bar{Z}_2^{n-1}\sqrt{2\tilde{\mu}}\sin\left(\tilde{\theta}\right) - \bar{Z}_1^{n-1}\frac{\cos\left(\tilde{\theta}\right)}{\sqrt{2\tilde{\mu}}}\right) dt, \\
& i_{\bar{Z}^{n-1}} d i_{\bar{Z}^1} d\bar{\boldsymbol{\beta}}_0 = -\left(\bar{Z}_1^{n-1}\frac{\partial \bar{Z}_1^1}{\partial \tilde{\mu}} + \bar{Z}_2^{n-1}\frac{\partial \bar{Z}_1^1}{\partial \tilde{\theta}}\right) dt.
\end{aligned}
\tag{3.73}
$$

Consequently, (3.72) reads:

$$
\frac{\partial S_n}{\partial \tilde{\mu}} - \bar{Z}_2^{n-1} + \psi_1^n\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right) - \tilde{\alpha}_n^1 = 0, \tag{3.74}
$$

$$
\frac{\partial S_n}{\partial \tilde{\theta}} + \bar{Z}_1^{n-1} + \psi_2^n\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right) - \tilde{\alpha}_n^2 = 0, \tag{3.75}
$$

and

$$
\begin{aligned}
& \bar{Z}_1^n + \frac{\partial S_n}{\partial t} - \tilde{\alpha}_n^3 - \phi_{n-1}\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right), t\right) \\
& - \left(\bar{Z}_2^{n-1}\sqrt{2\tilde{\mu}}\sin\left(\tilde{\theta}\right) - \bar{Z}_1^{n-1}\frac{\cos\left(\tilde{\theta}\right)}{\sqrt{2\tilde{\mu}}}\right)\frac{\partial \phi_0}{\partial r}\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right), t\right) \\
& + \psi_3^n\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right) - \left(\bar{Z}_1^{n-1}\frac{\partial \bar{Z}_1^1}{\partial \tilde{\mu}} + \bar{Z}_2^{n-1}\frac{\partial \bar{Z}_1^1}{\partial \tilde{\theta}}\right) = 0
\end{aligned}
\tag{3.76}
$$

Since $\mathcal{C}_{2\pi}^{\infty} = \mathcal{D} \oplus \mathcal{R}$, we make the following decompositions:

$$
S_n = S_n^{\mathcal{D}} + S_n^{\mathcal{R}}, \tag{3.77}
$$

$$
\bar{Z}_2^{n-1} = \bar{Z}_2^{n-1,\mathcal{D}} + \bar{Z}_2^{n-1,\mathcal{R}}, \tag{3.78}
$$

$$
\psi_2^n\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right) = \psi_2^{n,\mathcal{D}}\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right) + \psi_2^{n,\mathcal{R}}\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right). \tag{3.79}
$$

Setting $\tilde{\alpha}_n^2 = 0$ in (3.75) implies

$$
\frac{\partial S_n}{\partial \tilde{\theta}} + \bar{Z}_1^{n-1} + \psi_2^n\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right) = 0, \tag{3.80}
$$

and consequently we set:

$$
\frac{\partial S_n^{\mathcal{R}}}{\partial \tilde{\theta}} = -\bar{Z}_1^{n-1,\mathcal{R}} - \psi_2^{n,\mathcal{R}}\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right), \tag{3.81}
$$

$$
\bar{Z}_1^{n-1,\mathcal{D}} = -\psi_2^{n,\mathcal{D}}\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right). \tag{3.82}
$$

Since $\bar{Z}_1^{n-1,\mathcal{R}} + \psi_2^{n,\mathcal{R}}\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right) \in \mathcal{R}$, equation (3.81) has a solution in $\mathcal{R}$.

Afterwards, setting $S_n^{\mathcal{D}} = 0$,

$$\bar{Z}_2^{n-1} = \frac{\partial S_n^{\mathcal{R}}}{\partial \tilde{\mu}} - \psi_1^{n,\mathcal{R}}\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right), \tag{3.83}$$

$$\tilde{\alpha}_n^1 = \psi_1^{n,\mathcal{D}}\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right), \tag{3.84}$$

solve equation (3.74).

Finally, let ϱ_n be the function defined by

$$\begin{aligned}\varrho_n\left(\tilde{\mu}, \tilde{\theta}, t\right) = &-\frac{\partial S_n}{\partial t} + \phi_{n-1}\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right), t\right) \\ &+ \left(\bar{Z}_2^{n-1}\sqrt{2\tilde{\mu}}\sin\left(\tilde{\theta}\right) - \bar{Z}_1^{n-1}\frac{\cos\left(\tilde{\theta}\right)}{\sqrt{2\tilde{\mu}}}\right)\frac{\partial \phi_0}{\partial r}\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right), t\right) \\ &- \psi_3^n\left(\bar{\boldsymbol{Z}}^1, \ldots, \bar{\boldsymbol{Z}}^{n-2}\right) - \left(\bar{Z}_1^{n-1}\frac{\partial \bar{Z}_1^1}{\partial \tilde{\mu}} + \bar{Z}_2^{n-1}\frac{\partial \bar{Z}_1^1}{\partial \tilde{\theta}}\right)\end{aligned} \tag{3.85}$$

Then, equation (3.86) reads:

$$\bar{Z}_1^n - \tilde{\alpha}_n^3 - \varrho_n = 0. \tag{3.86}$$

Setting

$$\bar{Z}_1^{n,\mathcal{R}} = \varrho_n^{\mathcal{R}}, \tag{3.87}$$

$$\alpha_n^3 = -\varrho_n^{\mathcal{D}} + \bar{Z}_1^{n,\mathcal{D}} \tag{3.88}$$

remove the $\tilde{\theta}$ dependency in $\tilde{\alpha}_n^3$. This ends the induction step and the proof of Theorem 3.5. □

3.4 Proof of Theorem 1.1

Let $\boldsymbol{\mathcal{L}}_\varepsilon$ and $\tilde{\boldsymbol{\alpha}}_\varepsilon$ be the Lie change of coordinates and the normal form of $\boldsymbol{\gamma}^\varepsilon$ constructed in the proof of Theorem 3.3. According to Theorem (3.2) the expression of the particle distribution in the Lie coordinate system is given by:

$$\frac{\partial \tilde{f}_\varepsilon}{\partial t} + \left(\frac{\partial \tilde{\alpha}_\varepsilon^3}{\partial \tilde{\mu}} - \frac{\partial \tilde{\alpha}_\varepsilon^1}{\partial t}\right)\frac{\partial \tilde{f}_\varepsilon}{\partial t} = 0. \tag{3.89}$$

Setting

$$a_\varepsilon\left(\tilde{\mu}, t\right) = \left(\frac{\partial \tilde{\alpha}_\varepsilon^3}{\partial \tilde{\mu}}\left(\tilde{\mu}, t\right) - \frac{\partial \tilde{\alpha}_\varepsilon^1}{\partial t}\left(\tilde{\mu}, t\right)\right) \tag{3.90}$$

yields formula (1.31). Moreover, the Hilbert expansion of a_ε is given by

$$a_\varepsilon(\tilde{\mu},t)=\frac{1}{\varepsilon}\sum_{n\geq 0}\left(\frac{\partial\tilde{\alpha}_n^3}{\partial\tilde{\mu}}(\tilde{\mu},t)-\frac{\partial\tilde{\alpha}_n^1}{\partial t}(\tilde{\mu},t)\right)\varepsilon^n. \tag{3.91}$$

According to formula 3.40, the first term of this Hilbert expansion is given by

$$a_0(\tilde{\mu},t)=-1, \tag{3.92}$$

and according to formula (3.42), the second term of the Hilbert expansion is given by

$$a_1(\tilde{\mu},t)=\frac{1}{2\pi\sqrt{2\tilde{\mu}}}\int_{-\pi}^{\pi}\cos\left(\tilde{\theta}\right)E_0\left(\sqrt{2\tilde{\mu}}\cos\left(\tilde{\theta}\right),t\right)d\tilde{\theta}. \tag{3.93}$$

Formulas (3.92) and (3.93) yield formula (1.38).

The Poisson equation expressed in the (r,v_r,t) coordinate system is given by (1.2) and the charge density by (1.3). In order to solve the Vlasov Equation (3.89) we need to express the charge density ρ_ε in terms of the particle density expressed in the Lie coordinate system. Let $\bar{f}_\varepsilon$ the particle density expressed in the (μ,θ) coordinate system; i.e.,

$$\bar{f}_\varepsilon(\mu,\theta,t)=f_\varepsilon\left(\mathfrak{Pol}^{-1}(\mu,\theta),t\right),\ \text{ or equivalently} \tag{3.94}$$

$$f_\varepsilon(r,v_r,t)=\bar{f}_\varepsilon\left(\mathfrak{Pol}(r,v_r),t\right). \tag{3.95}$$

Then, the charge density ρ_ε, given by (1.3), can be rewritten as follow:

$$\begin{aligned}\rho_\varepsilon(t,r)&=\int_{\mathbb{R}}f_\varepsilon\left(r,v_r',t\right)dv_r'\\&=\int_{\mathbb{R}^2}f_\varepsilon\left(r',v_r',t\right)\delta\left(r-r'\right)dr'dv_r'\\&=\int_{\mathbb{R}_+\times]-\pi,\pi]}\bar{f}_\varepsilon\left(\mu',\theta',t\right)h_r\left(\mu',\theta'\right)d\mu'd\theta',\end{aligned}$$

where $h_r=h_r(\mu',\theta')$ is defined by

$$h_r\left(\mu',\theta'\right)=\delta\left(r-\sqrt{2\mu'}\cos\left(\theta'\right)\right). \tag{3.96}$$

Let $\tilde{f}_\varepsilon$ the particle density expressed in the $\left(\tilde{\mu},\tilde{\theta},t\right)$ coordinate system; i.e.,

$$\tilde{f}_\varepsilon\left(\tilde{\mu},\tilde{\theta},t\right)=\bar{f}_\varepsilon\left(\boldsymbol{\mathcal{L}}_\varepsilon^{-1}\left(\tilde{\mu},\tilde{\theta},t\right)\right),\ \text{ or equivalently} \tag{3.97}$$

$$\bar{f}_\varepsilon(\mu,\theta,t)=\tilde{f}_\varepsilon\left(\boldsymbol{\mathcal{L}}_\varepsilon(\mu,\theta,t)\right), \tag{3.98}$$

$\mathbf{D}_\varepsilon^t=\boldsymbol{\mathcal{L}}_\varepsilon(\mathbb{R}_+\times]-\pi,\pi],t)$ and $\left|\mathcal{J}_{\boldsymbol{\mathcal{L}}_\varepsilon^{-1}}\left(\tilde{\mu}',\tilde{\theta}',t'\right)\right|$ the jacobian associated with $\boldsymbol{\mathcal{L}}_\varepsilon^{-1}$. Then the charge density can be rewritten as follow:

$$\begin{aligned}\rho_\varepsilon(t,r)&=\int_{\boldsymbol{\mathcal{L}}_\varepsilon^{-1}(\mathbf{D}_\varepsilon^t)}\bar{f}_\varepsilon\left(\mu',\theta',t\right)h_r\left(\mu',\theta'\right)d\mu'd\theta'\\&=\int_{\mathbf{D}_\varepsilon^t}\tilde{f}_\varepsilon\left(\tilde{\mu}',\tilde{\theta}',t\right)h_r\left(\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{L}}_\varepsilon^{-1}\left(\tilde{\mu}',\tilde{\theta}',t\right)\right)\left|\mathcal{J}_{\boldsymbol{\mathcal{L}}_\varepsilon^{-1}}\left(\tilde{\mu}',\tilde{\theta}',t\right)\right|d\tilde{\mu}'d\tilde{\theta}'.\end{aligned}$$

Finally, Lemma 3.1 and Theorem 3.4 yields that:

$$\begin{aligned}\bar{\mathbf{Z}}^{1}(\mu,\theta,t) &= (\phi_0\left(\sqrt{2\mu}\cos(\theta),t\right) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_0\left(\sqrt{2\mu}\cos(\theta),t\right)d\theta)\partial_\mu \\ &\quad + (\frac{1}{\sqrt{2\mu}}\int_0^{\theta}\cos(\theta')\,E_0\left(\sqrt{2\mu}\cos(\theta'),t\right)d\theta' \\ &\quad - \frac{\theta}{2\pi\sqrt{2\mu}}\int_{-\pi}^{\pi}\cos(\theta')\,E_0\left(\sqrt{2\mu}\cos(\theta'),t\right)d\theta')\partial_\theta.\end{aligned} \tag{3.99}$$

Applying formulas (3.6) and (3.7) and truncating at the second order yields formulas (1.36) and (1.37). This ends the proof of Theorem 1.1.

3.5 Truncated models and some remarks about their efficiency

As we saw in the previous Subsection, for a given $N \in \mathbb{N}^\star$ the vector fields $\bar{\mathbf{Z}}_1,\ldots,\bar{\mathbf{Z}}_N$ allow us to construct the N first terms $\tilde{\boldsymbol{\alpha}}_0,\ldots,\tilde{\boldsymbol{\alpha}}_N$ of the normal form $\tilde{\boldsymbol{\alpha}}_\varepsilon$. Hence, defining the partial Lie change of coordinates of order N by

$$\boldsymbol{\mathcal{L}}_\varepsilon^N = \bar{\varphi}^N_{\varepsilon^N}\circ\ldots\circ\bar{\varphi}^1_{\varepsilon^1} \tag{3.100}$$

and making the change of coordinates $\left(\tilde{\mu},\tilde{\theta},t\right) = \boldsymbol{\mathcal{L}}_\varepsilon^N(\mu,\theta,t)$ lead to a differential one form $\tilde{\boldsymbol{\alpha}}_\varepsilon^{T,N} \in \left[\tilde{\boldsymbol{\beta}}^\varepsilon\right]$ which is up to order N under the normal form; i.e.,

$$\tilde{\boldsymbol{\alpha}}_\varepsilon^{T,N}\left(\tilde{\mu},\tilde{\theta},t\right) = \frac{1}{\varepsilon}\left(\sum_{n=0}^{N}\tilde{\boldsymbol{\alpha}}_n\left(\tilde{\mu},\tilde{\theta},t\right)\right) + \mathcal{O}\left(\varepsilon^N\right). \tag{3.101}$$

Consequently, Proposition 2.1 and Theorem 2.2 yield that the characteristics associated with the Vlasov equation (1.1) expressed in the partial Lie coordinate system of order N are given by

$$\frac{\partial\tilde{\mathfrak{Mu}}^{\varepsilon}_{T,N}}{\partial t}\left(\tilde{\mu},\tilde{\theta},t\right) = \mathcal{O}\left(\varepsilon^N\right), \tag{3.102}$$

$$\frac{\partial\tilde{\Theta}^{\varepsilon}_{T,N}}{\partial t}\left(\tilde{\mu},\tilde{\theta},t\right) = \frac{1}{\varepsilon}\left(\sum_{n=0}^{N}a_n\left(\tilde{\mathfrak{Mu}}^{\varepsilon}_{T,N},t\right)\varepsilon^n\right) + \mathcal{O}\left(\varepsilon^N\right), \tag{3.103}$$

provided with the initial conditions $\tilde{\mathfrak{Mu}}^{\varepsilon}_{T,N}\left(\tilde{\mu},\tilde{\theta},0\right) = \tilde{\mu}$ and $\tilde{\Theta}^{\varepsilon}_{T,N}\left(\tilde{\mu},\tilde{\theta},0\right) = \tilde{\theta}$.

Remark 3.2. *Notice that the reminders (the $\mathcal{O}\left(\varepsilon^N\right)$) depend to $\tilde{\mu}$, $\tilde{\theta}$ and t and that they are evaluated at the characteristics. By construction the vector fields $\bar{\mathbf{Z}}_1,\ldots,\bar{\mathbf{Z}}_N$ are 2π periodic with respect to θ. Consequently we can easily deduce that the first component of $\boldsymbol{\mathcal{L}}_\varepsilon^N$ is 2π periodic with respect to θ and that the second component satisfies*

$$\left(\boldsymbol{\mathcal{L}}_\varepsilon^N\right)_2\left(\tilde{\mu},\tilde{\theta}+2\pi,t\right) = \left(\boldsymbol{\mathcal{L}}_\varepsilon^N\right)_2\left(\tilde{\mu},\tilde{\theta},t\right) + 2\pi. \tag{3.104}$$

On the other hand, let $\boldsymbol{\tau}_H^\varepsilon$ be the vector field whose principal part in the (r, v_r, t) coordinate system is given by (2.4) *(with $G = H_\varepsilon$). Then, its expression in the polar coordinate system (μ, θ, t) is given by*

$$\begin{aligned}\bar{\mathbf{X}}_H^\varepsilon(\mu, \theta, t) = &\sqrt{2\mu}\sin(\theta)\, E^\varepsilon\left(\sqrt{2\mu}\cos(\theta), t\right)\partial_\mu \\ &+ \left(-\frac{1}{\varepsilon} + \frac{\cos(\theta)}{\sqrt{2\mu}}E^\varepsilon\left(\sqrt{2\mu}\cos(\theta), t\right)\right)\partial_\mu + \partial_t\end{aligned} \tag{3.105}$$

and it is consequently 2π periodic. Hence, the expression of $\boldsymbol{\tau}_H^\varepsilon$ in the $\left(\tilde{\mu}, \tilde{\theta}, t\right)$ coordinate system is 2π periodic with respect to θ. This implies that the $\mathcal{O}\left(\varepsilon^N\right)$ in (3.102)-(3.103) *are 2π periodic with respect to θ and consequently bounded with respect to this variable.*

Remark 3.3. *Since we deal with confined beams; i.e., the initial condition f_0 is chosen in such a way that the beam is bounded, the characteristic $\mathfrak{Mu}^\varepsilon$, which corresponds for a given particle to the evolution of the half of the square of the modulus between the origin and the particle position in the phase space, is bounded. Hence if we observe the evolution of the beam up to a given time $T \in (0, +\infty)$, the usual change of coordinate rules for the characteristics yield that $\tilde{\mathfrak{Mu}}_{T,N}^\varepsilon$ is also bounded for $t \in [0, T]$. Finally, since the reminders of* (3.102)-(3.103) *are 2π periodic with respect to $\tilde{\theta}$ and since $\tilde{\mathfrak{Mu}}_{T,N}^\varepsilon$ is bounded for $t \in [0, T]$ we obtain for any positive real number ν and for any $\varepsilon \in (0, \nu)$ an estimation $\left|\mathcal{O}\left(\varepsilon^N\right)\right| \leq C_N(T, \nu)\varepsilon^N$ for the reminders. Integrating these estimations yields error terms bounded by $C_N(T, \nu)\varepsilon^N T$.*

Remark 3.4. *$\boldsymbol{\mathcal{L}}_\varepsilon^N$ admits the following expansion in power of ε:*

$$\boldsymbol{\mathcal{L}}_\varepsilon^N = \left(\sum_{k=0}^{N}\varepsilon^k\left(\sum_{n_1+2n_2+\ldots+kn_k=k}\frac{\left(\bar{\mathbf{Z}}^1\right)^{n_1}\ldots\left(\bar{\mathbf{Z}}^k\right)^{n_k}}{n_1!\ldots n_k!}\right)(id)\right) + \mathcal{O}\left(\varepsilon^{N+1}\right). \tag{3.106}$$

Hence, the partial Lie change of coordinates $\boldsymbol{\mathcal{L}}_\varepsilon^N$ is an approximation of order $N+1$ of the Lie change of coordinates. Moreover, since the change of coordinates $\left(\tilde{\mu}, \tilde{\theta}, t\right) = \boldsymbol{\mathcal{L}}_\varepsilon^N(\mu, \theta, t)$ produces a $\mathcal{O}\left(\varepsilon^N\right)$ error term in the right hand side of (3.102)-(3.103)*, it produces a $\mathcal{O}\left(\varepsilon^N\right)$ error term in the characteristics. Hence, for numerical simulations it is sufficient to truncate* (3.106) *at order N. That is what we do in our simulations for $N = 1$.*

Remark 3.5. *As a consequence of the previous Remarks and since approximation* (1.46) *is obtained by making the change of coordinates $\boldsymbol{\mathcal{L}}_\varepsilon^1$, the error term in the characteristics is bounded by $C_1(\nu, T)\varepsilon T$ for any positive real numbers T and ν, and for any $\varepsilon \in (0, \nu)$ and $t \in (0, T)$. Hence, for small time T of simulation the accuracy is of order ε. For longer times the accuracy is rather* 1*. Nevertheless, we will observe numerically in Subsection 5 that for longer times of simulation the dynamics (fast rotation+slow filamentation) characterizing the evolution of the shape of the beam is close, but that the filaments are longer and wider. We will give more explanations in Subsection 5.*

4 Description of the numerical method

In this section, we will describe the PIC method that we will use in order to simulate equations (1.46)-(1.50) with the initial condition

$$f_0(r, v_r) = \frac{n_0}{\sqrt{2\pi} v_{th}} \exp\left(-\frac{v_r^2}{2v_{th}^2}\right) \chi_{[-0,75;0,75]}(r). \tag{4.1}$$

As usual in a PIC method, $\tilde{f}_\varepsilon$ will be approximated by the following Dirac mass sum:

$$\tilde{f}_\varepsilon^N\left(\tilde{\mu}, \tilde{\theta}, t\right) = \sum_{k=1}^{N} \omega_k \delta\left(\tilde{\mu} - \tilde{\mathfrak{Mu}}_k^\varepsilon(t)\right) \delta\left(\tilde{\theta} - \tilde{\Theta}_k^\varepsilon(t)\right) \tag{4.2}$$

where $\left(\tilde{\mathfrak{Mu}}_k^\varepsilon(t), \tilde{\Theta}_k^\varepsilon(t)\right)$ is the position in the $\left(\tilde{\mu}, \tilde{\theta}, t\right)$ coordinate system of macro-particle k which moves along a characteristic curve of the first order PDE (1.46). Hence the job is reduced to compute the macro-particle positions $\left(\tilde{\mathfrak{Mu}}_k^{\varepsilon,l+1}, \tilde{\Theta}_k^{\varepsilon,l+1}\right)$ at time $t_{l+1} = t_l + \Delta t$ from their positions $\left(\tilde{\mathfrak{Mu}}_k^{\varepsilon,l}, \tilde{\Theta}_k^{\varepsilon,l}\right)$ at time t_l, knowing they are solutions to

$$\frac{d\tilde{\mathfrak{Mu}}_k^\varepsilon}{dt}(t) = 0, \qquad \tilde{\mathfrak{Mu}}_k^\varepsilon(t_l) = \tilde{\mathfrak{Mu}}_k^{\varepsilon,l}, \tag{4.3}$$

$$\frac{d\tilde{\Theta}_k^\varepsilon}{dt}(t) = -\frac{1}{\varepsilon} + \frac{1}{2\pi\sqrt{2\tilde{\mathfrak{Mu}}_k^\varepsilon(t)}} \int_{-\pi}^{\pi} \cos\left(\tilde{\theta}'\right) E_\varepsilon\left(\sqrt{2\tilde{\mathfrak{Mu}}_k^\varepsilon(t)} \cos\left(\tilde{\theta}'\right), t\right) d\tilde{\theta}', \tag{4.4}$$

$$\tilde{\Theta}_k^\varepsilon(t_l) = \tilde{\Theta}_k^{\varepsilon,l}. \tag{4.5}$$

According to (4.3), for each $t \in \mathbb{R}_+$ and for each $k \in \{1, \ldots, N\}$, $\tilde{\mathfrak{Mu}}_k^\varepsilon(t) = \tilde{\mu}_k^0$ and the job is also reduced to integrate for each time step the equation

$$\frac{d\tilde{\Theta}_k^\varepsilon}{dt}(t) = -\frac{1}{\varepsilon} + \frac{1}{2\pi\sqrt{2\tilde{\mu}_k^0}} \int_{-\pi}^{\pi} \cos\left(\tilde{\theta}'\right) E_\varepsilon\left(\sqrt{2\tilde{\mu}_k^0} \cos\left(\tilde{\theta}'\right), t\right) d\tilde{\theta}', \tag{4.6}$$

$$\tilde{\Theta}_k^\varepsilon(t_l) = \tilde{\Theta}_k^{\varepsilon,l}. \tag{4.7}$$

Notice also that as $\theta \mapsto E_\varepsilon\left(\sqrt{2\tilde{\mu}} \cos\left(\tilde{\theta}\right), t\right)$ is an even function, the above integral can be replaced by

$$4\int_0^{\frac{\pi}{2}} \cos\left(\tilde{\theta}'\right) E_\varepsilon\left(\sqrt{2\tilde{\mu}} \cos\left(\tilde{\theta}'\right), t\right) d\tilde{\theta}'. \tag{4.8}$$

The first step of the computation of $\tilde{\Theta}_k^{\varepsilon,l+1}$ consists in replacing the integral above by p-node quadrature formula. As we approximate the integral of a periodic function over one period, the trapezoidal rule is optimal and will yield very accurate results for as few quadrature points as are needed to resolve the oscillations of the function.

Then, the equation for $\tilde{\Theta}_k^{\varepsilon,l+1}$ become

$$\frac{d\tilde{\Theta}_k^{\varepsilon}}{dt}(t) = -\frac{1}{\varepsilon} + \frac{2}{\pi\sqrt{2\tilde{\mu}_k^0}}\sum_{m=1}^{p}\Lambda_m \cos(\sigma_m)\, E_\varepsilon\left(\sqrt{2\tilde{\mu}_k^0}\cos(\sigma_m), t\right). \tag{4.9}$$

$$\tilde{\Theta}_k^{\varepsilon}(t_l) = \tilde{\Theta}_k^{\varepsilon,l}, \tag{4.10}$$

where $(\sigma_m)_{m=0}^{p}$ is a grid of $[0, \frac{\pi}{2}]$.

4.1 Expression of the initial condition in the Lie coordinates

The first step consists to replace the initial condition (4.1) by

$$f_0^N(r, v_r) = \sum_{k=1}^{N}\omega_k \delta\left(r - R_k^0\right)\delta\left(v_r - V_{r,k}^0\right), \tag{4.11}$$

where $(R_k^0)_{1\leq k\leq N}$ are uniformly distributed in $[-0,75; 0,75]$ and $(V_{r,k}^0)_{1\leq k\leq N}$ are normally distributed.

Using the following expression for θ

$$\theta = \begin{cases} \arctan\left(\frac{v_r}{r}\right) \text{ if } r > 0 \\ \arctan\left(\frac{v_r}{r}\right) + \pi \text{ if } r < 0 \text{ and } v_r \geq 0 \\ \arctan\left(\frac{v_r}{r}\right) - \pi \text{ if } r < 0 \text{ and } v_r < 0 \\ \frac{\pi}{2} \text{ if } r = 0 \text{ and } v_r > 0 \\ -\frac{\pi}{2} \text{ if } r = 0 \text{ and } v_r < 0 \end{cases} \tag{4.12}$$

and formula (1.14) for μ (Notice that formula (4.12) works only for $\mu \neq 0$. If $\mu = 0$ we set $\theta = 0$) we obtain the expression of the initial condition in the (μ, θ, t) coordinate system

$$\bar{f}_0^N(\mu, \theta) = \sum_{k=1}^{N}\omega_k \delta\left(\mu - \mathfrak{Mu}_k^0\right)\delta\left(\theta - \Theta_k^0\right). \tag{4.13}$$

Finally, using for each $1 \leq k \leq N$ the first order approximation (1.39)-(1.40) of the Lie change of coordinates, we obtain:

$$\begin{aligned} \tilde{\mu}_k^0 &= \mathfrak{Mu}_k^0, \\ \tilde{\Theta}_k^0 &= \Theta_k^0. \end{aligned} \tag{4.14}$$

Consequently, the expression of initial condition in the Lie coordinate system is given by:

$$\tilde{f}_0^N\left(\tilde{\mu}, \tilde{\theta}\right) = \sum_{k=1}^{N}\omega_k \delta\left(\tilde{\mu} - \tilde{\mu}_k^0\right)\delta\left(\tilde{\theta} - \tilde{\Theta}_k^0\right). \tag{4.15}$$

4.2 Numerical Resolution of (1.47)

Because of the form of the right hand side in (4.9) all along the algorithm, we need to compute values of the electric field E_ε generated by a given macro-particle distribution

$$\left(\tilde{\mu}_k^0, \tilde{\Theta}_k^\varepsilon(t)\right)_{k=1,\dots,N}.$$

Firstly, in order to solve (1.47) on $[-L, L]$ (L will be precise afterwards) we will proceed as follow. Injecting (4.2) in the right hand side of (1.47), and denoting by ρ_ε^N the yielding expression, we obtain:

$$\rho_\varepsilon^N(t,r) = \sum_{k=1}^{N} \omega_k \delta\left(r - \sqrt{2\tilde{\mu}_k^0} \cos\left(\tilde{\Theta}_k^\varepsilon(t)\right)\right). \tag{4.16}$$

Now, let $(r_k)_{k=0,\dots,m_P}$ be a uniform one-dimensional mesh of $[0, L]$. In order to obtain an expression of the right hand side of (1.47) on the grid, we will regularize (4.16) with first order spline

$$\rho_\varepsilon^h(t,r) = \sum_{k=1}^{N} \omega_k S^1\left(r - \sqrt{2\tilde{\mu}_k^0} \cos\left(\tilde{\Theta}_k^\varepsilon(t)\right)\right). \tag{4.17}$$

Afterwards, solving

$$\frac{\partial}{\partial r} r E_\varepsilon = r \rho_\varepsilon^h \tag{4.18}$$

on $(r_k)_{k=0,\dots,m_P}$ by integrating this equation with the trapezoidal rule yields the expression of the electric field E_ε on the grid. We denote by $\left(E_\varepsilon^k\right)_{k=1,\dots,m_p}$ these values. Notice that according to (1.4), $E_\varepsilon^0 = 0$. On the other hand, using the fact that E_ε is even we obtain the following expression for the electric field on $[-L, L]$:

$$E_\varepsilon^h(r,t) = dr_P \sum_{k=0}^{m_P} E_\varepsilon^k \left(S^1(r - r_k) - S^1(r + r_k)\right), \tag{4.19}$$

where $dr_P = (r_1 - r_0)/m_P$. At the end, in order to obtain the electric field E_ε on the macro particle $\left(\tilde{\mu}_k^0, \tilde{\Theta}_k^\varepsilon(t)\right)$ we just have to evaluate the above expression at $\sqrt{2\tilde{\mu}_k^0} \cos\left(\tilde{\Theta}_k^\varepsilon(t)\right)$.

4.3 Numerical Resolution of (4.9)-(4.10)

We solve (4.9)-(4.10) using the classical Runge-Kutta 4 method which gives the following scheme when applied to the computation of the approximation y^{l+1} of the value of y solution to $\frac{dy}{dt} = K(t,y)$ at time $t_l + \Delta t$ knowing its approximation y^l at time t_l :

$$\begin{aligned}
&t_{l,1} = t_l, \quad y^{l,1} = y^l, \\
&t_{l,2} = t_l + \frac{\Delta t}{2}, \quad y^{l,2} = y^l + \frac{1}{2} I^1, \quad \text{with } I^1 = \Delta t K(t_{l,1}, y^{l,1}), \\
&t_{l,3} = t_l + \frac{\Delta t}{2}, \quad y^{l,3} = y^l + \frac{1}{2} I^2, \quad \text{with } I^2 = \Delta t K(t_{l,2}, y^{l,2}), \\
&t_{l,4} = t_l + \Delta t, \quad y^{l,4} = y^l + I^3, \quad \text{with } I^3 = \Delta t K(t_{l,3}, y^{l,3}), \\
&\\
&y^{l+1} = y^l + \frac{1}{6} I^1 + \frac{1}{3} I^2 + \frac{1}{3} I^3 + \frac{1}{6} I^4, \text{ with } I^4 = \Delta t K(t_{l,4}, y^{l,4}).
\end{aligned} \tag{4.20}$$

Now, we will apply this scheme to our problem. So, the first step consists in computing $\left(\tilde{\Theta}_k^{\varepsilon,l,2}\right)_{1 \leq k \leq N}$ through:

$$\begin{aligned}
&\tilde{\Theta}_k^{\varepsilon,l,2} = \tilde{\Theta}_k^{\varepsilon,l} + \frac{1}{2} I^1, \text{ with} \\
&I^1 = \Delta t \left(-\frac{1}{\varepsilon} + \frac{2}{\pi \sqrt{2 \tilde{\mu}_k^0}} \sum_{m=1}^{p} \Lambda_m \cos(\sigma_m) E_\varepsilon \left(\sqrt{2 \tilde{\mu}_k^0} \cos(\sigma_m), t_{l,1} \right) \right),
\end{aligned} \tag{4.21}$$

where the value of $E_\varepsilon \left(\sqrt{2 \tilde{\mu}_k^0} \cos(\sigma_m), t_{l,1} \right)$ has been computed solving equation (1.47) associated with the particle distribution $\left(\Theta_k^{\varepsilon,l}\right)_{k=1,\ldots,N}$ by the procedure described in subsection (4.2).

The second step of the Runge-Kutta method consists in computing $\tilde{\Theta}_k^{\varepsilon,l,3}$ defined by

$$\begin{aligned}
&\tilde{\Theta}_k^{\varepsilon,l,3} = \tilde{\Theta}_k^{\varepsilon,l} + \frac{1}{2} I^2, \text{ with} \\
&I^2 = \Delta t \left(-\frac{1}{\varepsilon} + \frac{2}{\pi \sqrt{2 \tilde{\mu}_k^0}} \sum_{m=1}^{p} \Lambda_m \cos(\sigma_m) E_\varepsilon \left(\sqrt{2 \tilde{\mu}_k^0} \cos(\sigma_m), t_{l,2} \right) \right),
\end{aligned} \tag{4.22}$$

where the value of $E_\varepsilon \left(\sqrt{2 \tilde{\mu}_k^0} \cos(\sigma_m), t_{l,2} \right)$ is computed as previously from the $\left(\Theta_k^{\varepsilon,l,2}\right)_{k=1,\ldots,N}$ particle distribution.

Then we compute

$$\begin{aligned}
&\tilde{\Theta}_k^{\varepsilon,l,4} = \tilde{\Theta}_k^{\varepsilon,l} + I^3, \text{ with} \\
&I^3 = \Delta t \left(-\frac{1}{\varepsilon} + \frac{2}{\pi \sqrt{2 \tilde{\mu}_k^0}} \sum_{m=1}^{p} \Lambda_m \cos(\sigma_m) E_\varepsilon \left(\sqrt{2 \tilde{\mu}_k^0} \cos(\sigma_m), t_{l,3} \right) \right),
\end{aligned} \tag{4.23}$$

where $E_\varepsilon \left(\sqrt{2 \tilde{\mu}_k^0} \cos(\sigma_m), t_{l,3} \right)$ is computed from particle positions $\left(\Theta_k^{\varepsilon,l,3}\right)_{k=1,\ldots,N}$.

Finally, $\tilde{\Theta}_k^{\varepsilon,l+1}$ is obtained by the following formula:

$$\begin{aligned}
&\tilde{\Theta}_k^{\varepsilon,l+1} = \tilde{\Theta}_k^{\varepsilon,l} + \frac{1}{6}I^1 + \frac{1}{3}I^2 + \frac{1}{3}I^3 + \frac{1}{6}I^4, \text{ with} \\
&I^4 = \Delta t \left(-\frac{1}{\varepsilon} + \frac{2}{\pi\sqrt{2\tilde{\mu}_k^0}} \sum_{m=1}^{p} \Lambda_m \cos(\sigma_m)\, E_\varepsilon \left(\sqrt{2\tilde{\mu}_k^0} \cos(\sigma_m), t_{l+1} \right) \right),
\end{aligned} \tag{4.24}$$

where I^1, I^2 and I^3 are defined above and where $E_\varepsilon \left(\sqrt{2\tilde{\mu}_k^0} \cos(\sigma_m), t_{l+1} \right)$ is computed from particle positions $\left(\Theta_k^{\varepsilon,l,4} \right)_{k=1,\dots,N}$.

4.4 Expression of the particle density in the (r, v_r, t) coordinate system

Finally, using the previous algorithm, when we come to the desired time $t_f = m_f \Delta t$ of simulation we need to go back in the (r, v_r, t) coordinate system. Firstly, we go back in the (μ, θ, t) coordinate system. Applying for each $1 \leq k \leq N$ the first order approximation (1.39)-(1.40) of the Lie change of coordinates we obtain:

$$\begin{aligned}
\mathfrak{Mu}_k^{\varepsilon,m_f} &= \tilde{\mu}_k^0, \\
\Theta_k^{\varepsilon,m_f} &= \tilde{\Theta}_k^{\varepsilon,m_f}.
\end{aligned} \tag{4.25}$$

Afterwards, using formula (1.15)-(1.16) we obtain the particle density expressed in the (r, v_r, t) coordinate system.

5 Numerical simulations

For the numerical simulations we take a thermal velocity $v_{th} = 0.0727518214392$, an initial mass density $n_0 = 1$, a number $N = 1 \cdot 10^4$ of macro particles, constant weights $\omega_k = \omega = \frac{1}{N}$ in 4.2, a 18-node composed trapezoidal quadrature formula for the computation of (4.8), $L = 1.5$ and $m_P = 128$ for the Poisson mesh, a small parameter $\varepsilon = 10^{-3}$, a time step $\Delta t = \varepsilon\sqrt{\varepsilon}$ and a Box-Muller method in order to generate the initial condition. As no analytical solution is available, we will compare our result with a standard PIC method (see [4]). The simulation results are given in figures 1, 2 and 3.

Remark 5.1. *From Figure 1 one can see the announced property of accuracy for small times of simulations.*

Remark 5.2. *From Figure 2 one can see the evolution of μ for two given particles: one close to the center of the beam and the other close to the extremity of the beam.*

Remark 5.3. *In Figure 3 we observe that for longer times of simulation the dynamics characterizing the evolution of the shape of the beam (fast rotation+slow filamentation) is close to the reference solution but that the filaments are longer and wider. The reason is the following: we have made first order truncations in the dynamical system giving the characteristics*

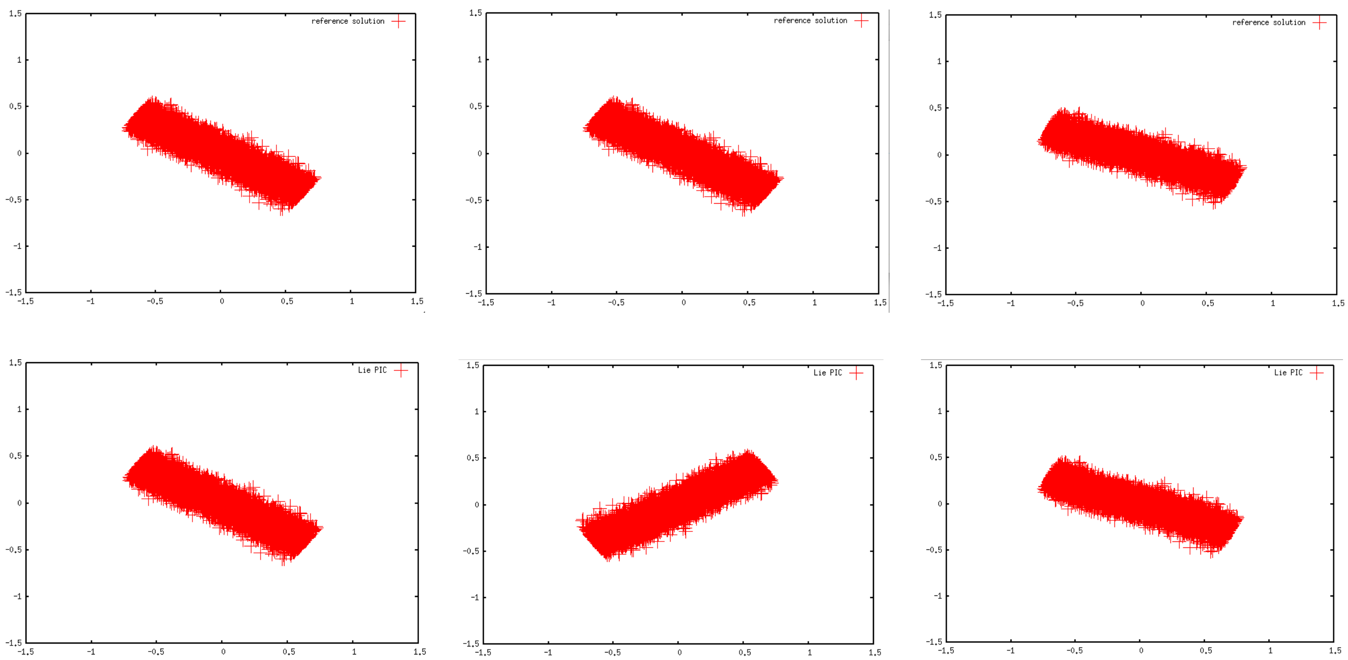

Figure 1: Beam simulation with an usual PIC method and a Lie PIC method for $\varepsilon = 0.001$. Left: beam at time 0.001, center: beam at time 0.1, right : beam at time 1. Top : Simulation provided with the usual PIC method, bottom: Simulation provided with the Lie PIC method.

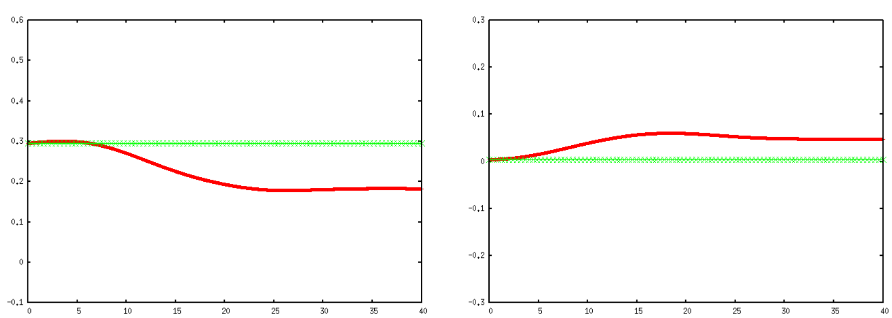

Figure 2: Evolution of μ up to time 40 with an usual PIC method and a Lie PIC method for $\varepsilon = 0.001$. Green: with the Lie PIC method, red: with the usual PIC method. Left: with initial condition $\mu = 0.2948404402060960$, right: with initial condition $\mu = 4.22461332489106316 \cdot 10^{-3}$

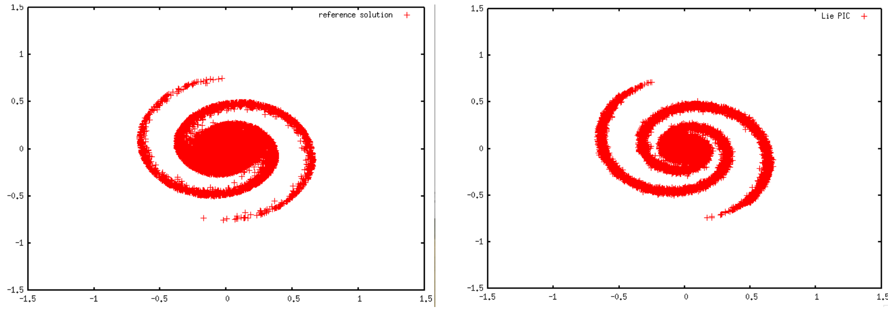

Figure 3: Beam simulation at time 35 with an usual PIC method and a Lie PIC method for $\varepsilon = 0.001$. Left: with an usual PIC method, right : with the Lie PIC method.

and in the change of coordinates. Within the framework of these first order truncations, the electric field is truncated at the first order and the square of the modulus between the origin and the particles position in the phase space become constant. The filamentation is due to the fact that the electric field is larger at the extremity of the beam as at the center. Moreover, without these truncations the particles of the extremity move toward the center of the beam. With these truncations the distance between the particles and the origin remain constant and consequently since the electric field is larger when one moves away from the center of the beam the phenomena of filamentation begins earlier and the filaments are wider.

6 Conclusions and perspectives

In this paper we have shown that we can adapt the geometrical techniques used for the derivation of the gyrokinetic coordinates to the case of a charged particle beam under the paraxial axisymetric approximation. In particular, these geometrical techniques are compatible with our way of doing the scaling. This paper is a first step in the application of these geometrical method, within our way to do the scaling (see Frénod & Sonnendrucker [5]), to the Vlasov Poisson equations modeling strongly magnetized plasmas. In particular, the derivation and the numerical simulations of these equations within our way to do the scaling, will allow us to compare the efficiently of this method with the other techniques of homogenization like the two scale methods. Probably, in order to eliminate a variable and to increase the time step, it will also be possible to combine the both methods. The numerical results are not only accurate but also promising, if one consider that they are only based on lowest order approximation of the electric field.

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References

[1] J. A. Brizard. Nonlinear gyrokinetic Vlasov equation for toroidally rotating axisymmetric tokamaks. *Physics of Plasmas*, 2(2):459–471, 1995.

[2] D. H. E. Dubin, J. A. Krommes, C. Oberman, and W. W. Lee. Nonlinear gyrokinetic equations. *Physics of Fluids*, XXVI(12):3524–3535, 1983.

[3] F. Filbet and E. Sonnendrücker. Modeling and numerical simulation of space charge dominated beams in the paraxial approximation. *Mathematical Models and Methods in Applied Sciences*, 16(5):763–791, 2006.

[4] E. Frénod, F. Salvarani, and E. Sonnendrücker. Long time simulation of a beam in a periodic focusing channel via a two-scale PIC-method. *Mathematical Models and Methods in Applied Sciences*, 19(2):175–197, 2009.

[5] E. Frénod and E. Sonnendrücker. The Finite Larmor Radius Approximation. *SIAM J. Math. Anal.*, 32(6):1227–1247, 2001.

[6] E. A. Frieman and L. Chen. Nonlinear gyrokinetic equations for low-frequency electromagnetic waves in general plasma equilibria. *Physics of Fluids*, 25(3):502–508, 1982.

[7] T. S. Hahm. Nonlinear gyrokinetic equations for tokamak microturbulence. *Physics of Fluids*, 31(9):2670–2673, 1988.

[8] T. S. Hahm, W. W. Lee, and A. Brizard. Nonlinear gyrokinetic theory for finite-beta plasmas. *Physics of Fluids*, 31(7):1940–1948, 1988.

[9] R. G. Littlejohn. A guiding center Hamiltonian: A new approach. *Journal of Mathematical Physics*, 20(12):2445–2458, 1979.

[10] R. G. Littlejohn. Hamiltonian formulation of guiding center motion. *Physics of Fluids*, 24(9):1730–1749, 1981.

[11] R. G. Littlejohn. Hamiltonian perturbation theory in noncanonical coordinates. *Journal of Mathematical Physics*, 23(5):742–747, 1982.

[12] Jerrold E. Marsden and Tudor S. Ratiu. *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*. Springer, 1999.

[13] K. R. Meyer and D. Hall, G. R.and Offin. *Introduction to hamiltonian dynamical systems and the N-Body Problem*, volume 90 of *Applied Mathematical Sciences*. Springer-Verlag, 1979.

[14] P. J. Olver. *Applications of Lie groups to differential equations*. Graduate Texts in Mathematics. Springer, 2000.

[15] F. I. Parra and P. J. Catto. Limitations of gyrokinetics on transport time scales. *Plasma Physics and Controlled Fusion*, 50(6):065014, 2008.

[16] F. I. Parra and P. J. Catto. Gyrokinetic equivalence. *Plasma Physics and Controlled Fusion*, 51(6):065002, 2009.

[17] F. I. Parra and P. J. Catto. Turbulent transport of toroidal angular momentum in low flow gyrokinetics. *Plasma Physics and Controlled Fusion*, 52(4):045004, 2010.

[18] H. Qin, R. H. Cohen, W. M. Nevins, and X. Q. Xu. General gyrokinetic equations for edge plasmas. *Contributions to Plasma Physics*, 46(7-9):477–489, 2006.